

Schubert calculus and cohomology of Lie groups.

Part II. Compact Lie groups

Haibao Duan*

Hua Loo-Keng Key Laboratory of Mathematics,
Institute of Mathematics, Chinese Academy of Sciences,
dhb@math.ac.cn

April 18, 2016

Abstract

Let G be a compact Lie group with a maximal torus T . Based on a Schubert presentation on the integral cohomology $H^*(G/T)$ of the flag manifold G/T [6] we have presented in [7] an explicit and unified construction of the integral cohomology $H^*(G)$ for all 1-connected Lie groups G . In this paper we extend this construction to the compact Lie groups.

2000 Mathematical Subject Classification: 55T10, 14M15

Key words and phrases: Lie groups; Cohomology, Schubert calculus

1 Introduction

The Lie group G under consideration is compact and connected; the coefficient for the cohomologies is either the ring \mathbb{Z} of integers, or one of the finite fields \mathbb{F}_p , unless otherwise stated.

For a maximal torus T on G the homogeneous space G/T is canonically a projective variety, called *the complete flag manifold* of the Lie group G . Based on a Schubert presentation on the integral cohomology ring of G/T [6, Theorem 1.2] we have presented in [7] a unified construction for the cohomologies of all 1-connected Lie groups. In this paper we extend this construction to the compact Lie groups. As applications of our general approach the integral cohomologies of the adjoint Lie groups $PSU(n)$, $PSp(n)$, PE_6 , PE_7 are determined, see Theorem 4.7, Theorem 4.12, as well as the historical remarks in Section 4.5.

The problem of computing the cohomologies of Lie groups was raised by Cartan in 1929. It is a focus of algebraic topology for the fundamental roles of Lie groups playing in geometry and topology, see [3, Chapter VI], [12, 18]. On the other hand, the classical Schubert calculus amounts to the determination of the cohomology rings of the flag manifolds G/T [23, p.331]. The present work, together with the companion ones [7, 8], completes our project to determine the integral cohomologies of all compact Lie groups G , as well as the structure of

*The author's research is supported by 973 Program 2011CB302400 and NSFC 11131008.

the mod p cohomology $H^*(G; \mathbb{F}_p)$ as a module over the Steenrod algebra \mathcal{A}_p , in the context of Schubert calculus.

We begin with a general procedure that reduces the construction and computation with the cohomology of a compact Lie group to that of the 1-connected ones. It is well known that all the 1-connected simple Lie groups G , together with their centers $\mathcal{Z}(G)$, are classified by the types Φ_G of their root systems showing in the following table, where $e \in G$ is the group unit:

Table 1.1. The types and centers of the 1-connected simple Lie groups

G	$SU(n)$	$Sp(n)$	$Spin(2n+1)$	$Spin(2n)$	G_2	F_4	E_6	E_7	E_8
Φ_G	A_{n-1}	B_n	C_n	D_n	G_2	F_4	E_6	E_7	E_8
$\mathcal{Z}(G)$	\mathbb{Z}_n	\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z}_4, n = 2k+1$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2, n = 2k$	$\{e\}$	$\{e\}$	\mathbb{Z}_3	\mathbb{Z}_2	$\{e\}$

In general, let G be a compact Lie group with $\mathcal{Z}_0(G)$ the identity component of the center of G , and with G' the commutator subgroup of G . Then the intersection $F = G' \cap \mathcal{Z}_0(G)$ is always a finite abelian group. The Cartan's classification on compact Lie groups states that ([21, Theorem 5.22])

Theorem 1.1. *The isomorphism type of a compact Lie group G is*

$$(1.1) \quad G = [G' \times \mathcal{Z}_0(G)] / F,$$

where F is embedded in the numerator group $G' \times \mathcal{Z}_0(G)$ as $\{(g, g^{-1}) \mid g \in F\}$.

Moreover, the commutator subgroup G' admits a canonical presentation as

$$(1.2) \quad G' = [G_1 \times \cdots \times G_k] / K,$$

where each G_t , $1 \leq t \leq k$, is one of the 1-connected simple Lie groups listed in Table 1.1, and where K is a subgroup of the finite group $\mathcal{Z}(G_1) \times \cdots \times \mathcal{Z}(G_k)$. \square

In views of (1.1) and (1.2) a Lie group G is called *semi-simple* if $\mathcal{Z}_0(G) = \{e\}$; *simple* if $\mathcal{Z}_0(G) = \{e\}$ and $k = 1$. Since the commutator subgroup G' is always semi-simple we shall call it the *semi-simple part* of G . Based on Theorem 1.1 we give a diffeomorphism classification of compact Lie groups in the following result. Let T^r be the r -dimensional torus group.

Theorem 1.2. *The diffeomorphism type of a compact Lie group G with semi-simple part G' is*

$$(1.3) \quad G \cong G' \times T^r, \quad r = \dim \mathcal{Z}_0(G).$$

Proof. Since G' is normal in G the quotient space $G/G' = \mathcal{Z}_0(G)/F$ has the structure of an abelian group isomorphic to the r -dimensional torus group T^r , $r = \dim \mathcal{Z}_0(G)$, and the quotient map $h : G \rightarrow G/G' = T^r$ is both a group homomorphism and a submersion with fiber G' .

Take a maximal torus T' on G' . By (1.1) a maximal torus on G is $T = [T' \times \mathcal{Z}_0(G)] / F$. Since the restriction $h|T : T \rightarrow T^r$ of h on T is a fiber bundle in torus groups there is a monomorphism $\sigma : T^r \rightarrow T$ so that the composition $(h|T) \circ \sigma$ is the identity on T^r . Since σ can be viewed as a section of h , one obtains the diffeomorphism (1.3) from the fact that the fiber of h is a group. \square

The unitary group $U(n)$ of order n may serve as the first example illustrating the subtle difference between the isomorphism and diffeomorphism types of a Lie group. As a group it is isomorphic to $[SU(n) \times S^1] / \mathbb{Z}_n$, while as a smooth manifold it is diffeomorphic to the product space $SU(n) \times S^1$.

Let G be a semi-simple Lie group whose center $\mathcal{Z}(G)$ contains the cyclic group \mathbb{Z}_q of order q . Consider the cyclic covering $c : G \rightarrow G/\mathbb{Z}_q$ of Lie groups. Since the classifying space $B\mathbb{Z}_q$ of the group \mathbb{Z}_q is the Eilenberg–MacLane space $K(\mathbb{Z}_q, 1)$ the classifying map $f_c : G/\mathbb{Z}_q \rightarrow B\mathbb{Z}_q$ of c defines a cohomology class $\iota \in H^1(G/\mathbb{Z}_q; \mathbb{F}_q)$, called *the characteristic class of the covering*.

On the other hand let \mathbb{Z}_q act on the circle group S^1 as the anti-clockwise rotation through the angle $2\pi/q$. Then the obvious group homomorphism

$$(1.4) \quad C : [G \times S^1] / \mathbb{Z}_q \rightarrow G/\mathbb{Z}_q$$

is an oriented cycle bundle on the quotient group G/\mathbb{Z}_q with Euler class $\omega = \beta_q(\iota) \in H^2(G/\mathbb{Z}_q)$, where β_q is the Bockstein homomorphism. The Gysin sequence of the circle bundle C then yields the exact sequence [17, P.143]

$$(1.5) \quad \cdots \rightarrow H^r(G/\mathbb{Z}_q) \xrightarrow{C^*} H^r(G \times S^1) \xrightarrow{\theta} H^{r-1}(G/\mathbb{Z}_q) \xrightarrow{\omega} H^{r+1}(G/\mathbb{Z}_q) \xrightarrow{C^*} \cdots$$

where ω denotes the homomorphism of taking product with the class ω , and where the space $[G \times S^1] / \mathbb{Z}_q$ has been replaced by its diffeomorphism type $G \times S^1$ by Theorem 1.2. Let $p : G \times S^1 \rightarrow S^1$ be the projection onto the second factor, and $\varepsilon \in H^1(S^1)$ the canonical orientation on S^1 . In what follows we set

$$(1.6) \quad \xi_1 := p^*(\varepsilon) \in H^1(G \times S^1).$$

Let $J(\omega)$ and $\langle \omega \rangle$ be respectively the subring and the ideal of $H^*(G/\mathbb{Z}_q)$ generated by ω . Write $H^*(G/\mathbb{Z}_q)_{\langle \omega \rangle}$ for the quotient ring $H^*(G/\mathbb{Z}_q) / \langle \omega \rangle$ with quotient map g . Then, in addition to the short exact sequence of rings

$$(1.7) \quad 0 \rightarrow \langle \omega \rangle \rightarrow H^*(G/\mathbb{Z}_q) \xrightarrow{g} H^*(G/\mathbb{Z}_q)_{\langle \omega \rangle} \rightarrow 0,$$

the cohomology $H^*(G/\mathbb{Z}_q)$ can be regarded as a module over its subring $J(\omega)$.

Theorem 1.3. *The induced map C^* fits into the exact sequence*

$$(1.8) \quad 0 \rightarrow H^*(G/\mathbb{Z}_q)_{\langle \omega \rangle} \xrightarrow{C^*} H^*(G \times S^1) \xrightarrow{\theta} H^*(G/\mathbb{Z}_q) \xrightarrow{\omega} \langle \omega \rangle \rightarrow 0$$

in which the homomorphism θ has the following properties

- i) $\theta(\xi_1) = q \in H^0(G/\mathbb{Z}_q)$;
- ii) $\theta(x \cup C^*(y)) = \theta(x) \cup y$ for $x \in H^*(G \times S^1)$ and $y \in H^*(G/\mathbb{Z}_q)$.

Moreover, if the map g in (1.7) admits a split homomorphism $j : H^*(G/\mathbb{Z}_q)_{\langle \omega \rangle} \rightarrow H^*(G/\mathbb{Z}_q)$, then the map

$$h : J(\omega) \otimes H^*(G/\mathbb{Z}_q)_{\langle \omega \rangle} \rightarrow H^*(G/\mathbb{Z}_q)$$

by $h(\omega^r \otimes x) = \omega^r \cup j(x)$ induces an isomorphism of $J(\omega)$ –modules

$$(1.9) \quad H^*(G/\mathbb{Z}_q) \cong \frac{J(\omega) \otimes H^*(G/\mathbb{Z}_q)_{\langle \omega \rangle}}{\langle \omega \cdot \text{Im } \theta \rangle}.$$

Proof. The sequence (1.8) is easily seen to be a compact form of (1.5), while the properties i) and ii) about the map θ are also known, see [14, Lemma 1].

The group $J(\omega)$ has the basis $\{1, \omega, \dots, \omega^r\}$ for some $r \geq 1$. Granted with the map j the sequence (1.7) is applicable to show that every $x \in H^*(G/\mathbb{Z}_q)$ admits an expansion of the form

$$x = j(a_0) + \omega \cup j(a_1) + \dots + \omega^r \cup j(a_r), \quad a_i \in H^*(G/\mathbb{Z}_q)_{\langle \omega \rangle}, \quad 0 \leq i \leq r.$$

That is the map h is surjective. To show the formula (1.9) consider an element $y = a_0 + \omega \otimes a_1 + \dots + \omega^r \otimes a_r \in \ker h$. One infers from $g \circ h(y) = 0$ that $a_0 = 0$. It follows now from $0 = h(y) = \omega \cdot (1 \cup j(a_1) + \dots + \omega^{r-1} \cup j(a_r))$ that

$$1 \cup j(a_1) + \dots + \omega^{r-1} \cup j(a_r) \in \text{Im } \theta$$

by (1.8). That is $\ker h = \langle \omega \cdot \text{Im } \theta \rangle$. \square

Granted with Theorems 1.2 and 1.3 we can clarify the main ideas of our approach. Firstly, Theorem 1.2 reduces the cohomology of a compact Lie group G to that of its semi-simple part G' by the Künneth formula $H^*(G) = H^*(G') \otimes H^*(T')$. Next, for a semi-simple Lie group G its universal covering $c : G_0 \rightarrow G$ can always be decomposed into a sequence of cyclic coverings

$$c : G_0 \xrightarrow{c_1} G_1 \xrightarrow{c_2} \dots \xrightarrow{c_k} G_k = G$$

in which the cohomology of the 1-connected Lie group G_0 is known [7, Theorem 1.9]. The cohomology $H^*(G)$ in question can be calculated, in principle, from the known one $H^*(G_0)$ by a repeatedly application of the exact sequence (1.8).

The remaining sections of the paper are so arranged. In Sections 2 and 3 we develop the formulae and the constructions required to implement above procedure. As applications the integral cohomology of the adjoint Lie groups $PSU(n), PSp(n), PE_6, PE_7$ are calculated in Section 4.

In addition, since the present work is a natural extension of the earlier one [7], we shall be free to adopt the notation and results developed in [7].

2 Spectral sequence of the fibration $G \rightarrow G/T$

Fix a maximal torus T on G , and consider the Leray–Serre spectral sequence $\{E_r^{*,*}(G), d_r\}$ of the corresponding fibration

$$(2.1) \quad T \hookrightarrow G \xrightarrow{\pi} G/T.$$

It is well known that its second page is the Koszul complex with

$$(2.2) \quad E_2^{*,*}(G) = H^*(G/T) \otimes H^*(T);$$

$$(2.3) \quad d_2(a \otimes t) = (\tau(t) \cup a) \otimes 1 \quad \text{for } a \in H^*(G/T), t \in H^1(T)$$

where $\tau : H^1(T) \rightarrow H^2(G/T)$ is the Borel transgression in the fibration (2.1).

Our calculation and construction with the exact sequence (1.8) actually take place on the third page of the spectral sequence. In order to access $E_3^{*,*}(G)$ we deduce a formula for the transgression τ in Theorem 2.4, and give a concise characterization for the factor ring $H^*(G/T)$ of $E_2^{*,*}(G)$ in Theorem 2.6. These results are essential in Section 3 for us to construct explicit generators of the ring $H^*(G)$ by certain polynomials in the Schubert classes on G/T .

2.1 A formula for the Borel transgression τ

In the diagram with top row the cohomology exact sequence of the pair (G, T)

$$0 \rightarrow H^1(G) \xrightarrow{i^*} H^1(T) \xrightarrow{\delta} H^2(G, T) \xrightarrow{j^*} H^2(G) \rightarrow \cdots$$

$$\searrow \tau \quad \cong \uparrow \pi^* \quad \quad \quad H^2(G/T)$$

the induced map π^* is an isomorphism by the 1-connectness of the pair (G, T) . The *Borel transgression* in the fibration (2.1) is $\tau = (\pi^*)^{-1} \circ \delta$ ([15, p.185]).

Lemma 2.1. *The diffeomorphism type of the flag manifold G/T depends only on the semi-simple part G' of the group G as*

$$(2.4) \quad G/T \cong \frac{G_1}{T_1} \times \cdots \times \frac{G_k}{T_k} \quad (\text{see (1.2)}),$$

where T_i is a maximal torus of the 1-connected simple Lie group G_i , $1 \leq i \leq k$.

The transgression τ fits into the following exact sequence, where $\text{Tor}(A)$ denotes the torsion subgroup of an abelian group A ,

$$(2.5) \quad 0 \rightarrow H^1(G) \xrightarrow{j^*} H^1(T) \xrightarrow{\tau} H^2(G/T) \xrightarrow{\pi^*} \text{Tor} H^2(G) \rightarrow 0.$$

Proof. Let T' be a maximal torus of the semi-simple part G' of G . By (1.1) a maximal torus of G is $T = [T' \times \mathcal{Z}_0(G)]/F$. The diffeomorphism (2.4) comes from the obvious relation $G/T = G'/T'$ and (1.2).

Since the second homotopy group of a Lie group is trivial, the homotopy exact sequence of π contains the free resolution of the fundamental group $\pi_1(G)$

$$0 \rightarrow \pi_2(G/T) \rightarrow \pi_1(T) \rightarrow \pi_1(G) \rightarrow 0.$$

Applying the co-functor $\text{Hom}(\cdot, \mathbb{Z})$ to this sequence, and using the Hurewicz isomorphisms $\pi_2(G/T) = H_2(G/T)$, $\pi_1(T) = H_1(T)$, $\pi_1(G) = H_1(G)$ to substitute for the relevant groups, one obtains (2.5). \square

By (1.3) we have $H^1(G) = H^1(T^r)$, $r = \dim \mathcal{Z}_0(G)$. By (2.5) the transgression τ annihilates the direct summand $H^1(T^r)$ of $H^1(T)$ and hence, depends only on the semi-simple part of G . For this reason we can assume below that the Lie group G under consideration is semi-simple.

Equip the Lie algebra $L(G)$ of G with an inner product (\cdot, \cdot) so that the adjoint representation acts as isometries on $L(G)$. Let $L(T) \subset L(G)$ be the *Cartan subalgebra* corresponding to the fixed maximal torus T on G , and fix a set $\Delta = \{\alpha_1, \dots, \alpha_n\} \subset L(T)$ of simple roots of G , where $n = \dim T$.

The Euclidean space $L(T)$ contains three distinguished lattices. Firstly, the set $\{\alpha_1, \dots, \alpha_n\}$ of simple roots generates the *root lattice* Λ_r of G . Next, the pre-image of the exponential map $\exp : L(T) \rightarrow T$ at the group unit $e \in T$ gives rise to the *unit lattice* $\Lambda_e := \exp^{-1}(e)$ of G . Finally, using simple roots one defines the set $\Omega = \{\phi_1, \dots, \phi_n\}$ of *fundamental dominant weights* of G by the formula $2(\phi_i, \phi_j)/(\alpha_j, \alpha_j) = \delta_{i,j}$ that generates the *weight lattice* Λ_ω of G .

Let $A = (b_{ij})_{n \times n}$, $b_{ij} = 2(a_i, \alpha_j)/(\alpha_j, \alpha_j)$, be the Cartan matrix of G , and let A^τ be the transpose of A . The following result can be found in [5, (3.4)].

Lemma 2.2. *On the space $L(T)$ one has $\Lambda_r \subseteq \Lambda_e \subseteq \Lambda_\omega$. In addition*

- i) the group G is 1-connected if and only if $\Lambda_r = \Lambda_e$;
- ii) the group G is adjoint if and only if $\Lambda_e = \Lambda_\omega$;
- iii) the basis Δ on Λ_r can be expressed by the basis Ω on Λ_ω by the formula

$$(\alpha_1, \dots, \alpha_n) = (\phi_1, \dots, \phi_n) \cdot A^\tau. \square$$

For a root $\alpha \in \Delta$ let $K(\alpha) \subset G$ be the subgroup with Lie algebra $l_\alpha \oplus L_\alpha$, where $l_\alpha \subset L(T)$ is the 1-dimensional subspace spanned by α , and $L_\alpha \subset L(G)$ is the root space (viewed as an oriented real 2-plane) belonging to the root α ([10, p.35]). Then the circle subgroup $S^1 = \exp(l_\alpha)$ is a maximal torus on $K(\alpha)$, while quotient space K_α/S^1 is diffeomorphic to the 2-dimensional sphere S^2 . Moreover, the inclusion $(K_\alpha, S^1) \subset (G, T)$ induces an embedding

$$(2.6) \quad s_\alpha : S^2 = K_\alpha/S^1 \rightarrow G/T$$

whose image is known as the *Schubert variety* associated to the root α [?]. By the basis theorem of Chevalley [2] the maps s_α with $\alpha \in \Delta$ represent a basis of the second homology $H_2(G/T)$. As a result if one lets $\omega_i \in H^2(G/T)$ be the Kronecker dual of the homology class represented by the map s_{α_i} , then

Lemma 2.3. *The set $\{\omega_1, \dots, \omega_n\}$ is a basis of the group $H^2(G/T)$. \square*

On the other hand let $\Theta = \{\theta_1, \dots, \theta_n\}$ be a basis for the unit lattice Λ_e . It defines n oriented circle subgroups on the maximal torus

$$(2.7) \quad \tilde{\theta}_i : S^1 = \mathbb{R}/\mathbb{Z} \rightarrow T, \quad \tilde{\theta}_i(t) := \exp(t\theta_i), \quad 1 \leq i \leq n,$$

that represent also a basis of the first homology $H_1(T)$. As result if we let $t_i \in H^1(T)$ be the class Kronecker dual to the map $\tilde{\theta}_i$, then

$$(2.8) \quad H^*(T) = \Lambda(t_1, \dots, t_n) \quad (\text{i.e. the exterior ring generated by } t_1, \dots, t_n).$$

Let $C(\Theta) = (c_{i,j})_{n \times n}$ be the matrix expressing the ordered basis Δ by the ordered basis Θ in view of the inclusion $\Lambda_r \subseteq \Lambda_e$ by Lemma 2.2. Namely, $(\alpha_1, \dots, \alpha_n) = (\theta_1, \dots, \theta_n) C(\Theta)$.

Theorem 2.4. *With respect to the basis (2.6) and (2.8) on the groups $H^2(G/T)$ and $H^1(T)$, the transgression τ is given by the formula*

$$(2.9) \quad (\tau(t_1), \dots, \tau(t_n)) = (\omega_1, \dots, \omega_n) C(\Theta).$$

Proof. Assume firstly that the group G is 1-connected. Then a basis Θ of the unit lattice $\Lambda_e = \Lambda_r$ can be taken to be $\Delta = \{\alpha_1, \dots, \alpha_n\}$. Since $C(\Theta)$ is then the identity matrix we are bound to show that $\tau(t_i) = \omega_i$, $1 \leq i \leq n$.

For each root $\alpha_i \in \Delta$ the inclusion $(K(\alpha_i), S^1) \subset (G, T)$ induces the following bundle map over s_{α_i} , where by the 1-connectness of the group G the group $K(\alpha_i)$ is isomorphic to the 3-sphere S^3 and π_i is the Hopf fibration over S^2 ,

$$\begin{array}{ccccccc} S^1 & \rightarrow & K(\alpha_i) = S^3 & \xrightarrow{\pi_i} & K(\alpha_i)/S^1 = S^2 \\ \tilde{\alpha}_1 \downarrow & & \downarrow & & \downarrow s_{\alpha_i} \\ T & \rightarrow & G & \xrightarrow{\pi} & G/T \end{array}$$

This indicates that in the homotopy exact sequence of π_i the connecting homomorphism ∂ satisfies $\partial[\iota_2] = [\iota_1]$, where ι_r is the identity on the r -sphere. By the naturality of ∂ one gets in the homotopy exact sequence of π that $\partial[s_{\alpha_i}] = [\tilde{\alpha}_i]$. This shows $\tau(t_i) = \omega_i$ as the map τ is dual to ∂ by the proof of Lemma 2.1.

In a general case let $d : (G_0, T_0) \rightarrow (G, T)$ be the universal covering of G with T_0 the maximal torus on G_0 corresponding to T . Then

$$\exp = d \circ \exp_0 : (L(G_0), L(T_0)) \rightarrow (G_0, T_0) \rightarrow (G, T).$$

where \exp (resp. \exp_0) is the exponential map of the group G (resp. G_0). It follows that, if we let $p(\Lambda_r, \Lambda_e) : T_0 = L(T_0)/\Lambda_r \rightarrow T = L(T_0)/\Lambda_e$ be the covering map induced by the inclusion $\Lambda_r \subset \Lambda_e$ of the lattices, then

$$(2.10) \quad d|_{T_0} = p(\Lambda_r, \Lambda_e) : T_0 \rightarrow T,$$

and the induced map $p(\Lambda_r, \Lambda_e)_*$ on $\pi_1(T_0)$ is

$$(2.11) \quad p(\Lambda_r, \Lambda_e)_*[\tilde{\alpha}_i] = c_{i,1}[\tilde{\theta}_1] + \cdots + c_{i,n}[\tilde{\theta}_n] \text{ with } C(\Theta) = (c_{ij})_{n \times n}.$$

On the other hand the restriction $d|_{T_0}$ fits in the commutative diagram

$$(2.12) \quad \begin{array}{ccc} \pi_2(G_0/T_0) & \xrightarrow[\cong]{\partial_0} & \pi_1(T_0) \\ \parallel & & \downarrow (d|_{T_0})_* \\ \pi_2(G/T) & \xrightarrow{\partial} & \pi_1(T) \end{array}$$

with ∂_0, ∂ the connecting homomorphisms in the homotopy exact sequences of the bundles $G_0 \rightarrow G_0/T_0, G \rightarrow G/T$, respectively, where the vertical identification on the left comes from (2.1). It follows that, for each root $\alpha_i \in \Delta$,

$$\begin{aligned} \partial[s_{\alpha_i}] &= (d|_{T_0})_* \circ \partial_0[s_{\alpha_i}] \text{ (by the diagram (2.12))} \\ &= (d|_{T_0})_*[\tilde{\alpha}_i] \text{ (by the proof of the previous case)} \\ &= p(\Lambda_r, \Lambda_e)_*([\tilde{\alpha}_i]) \text{ (by (2.10))}. \end{aligned}$$

The proof is completed by (2.11), and by the fact that the map τ is dual to ∂ . \square

Example 2.5. Formula (2.9) is ready to evaluate the transgression τ , hence the differential d_2 on $E_2^{*,*}(G)$. As examples we have by Lemma 2.2 that

- i) if the group G is 1-connected one can take $\Delta = \{\alpha_1, \dots, \alpha_n\}$ as a basis for the unit lattice Λ_e , and the transition matrix $C(\Theta)$ is the identity;
- ii) if the group G is of the adjoint type, then the set $\Omega = \{\phi_1, \dots, \phi_n\}$ of fundamental dominant weights is a basis of Λ_e , and the corresponding transition matrix $C(\Theta)$ from Λ_e to Λ_r is the transpose A^τ of the Cartan matrix A . \square

2.2 Schubert presentation of the ring $H^*(G/T)$

Turning to a concise presentation of the factor subring $H^*(G/T)$ of $E_2^{*,*}(G)$ assume that the rank of the semi-simple part of G is n , and let $\{\omega_1, \dots, \omega_n\}$ be the Schubert basis on $H^2(G/T)$. It is shown in [6, Theorem 1.2] that

Theorem 2.6. For each Lie group G there exist a set $\{y_1, \dots, y_m\}$ of Schubert classes on G/T with $\deg y_i > 2$, so that the set $\{\omega_1, \dots, \omega_n, y_1, \dots, y_m\}$ is a minimal system of generators of the integral cohomology ring $H^*(G/T)$.

With respect to these generators the ring $H^*(G/T)$ has the presentation

$$(2.13) \quad H^*(G/T) = \mathbb{Z}[\omega_1, \dots, \omega_n, y_1, \dots, y_m] / \langle h_i, f_j, g_j \rangle_{1 \leq i \leq k; 1 \leq j \leq m},$$

in which

- i) for each $1 \leq i \leq k$, $h_i \in \langle \omega_1, \dots, \omega_n \rangle$;
- ii) for each $1 \leq j \leq m$, the pair (f_j, g_j) of polynomials is related to the Schubert class y_j in the fashion

$$f_j = p_j y_j + \alpha_j, \quad g_j = y_j^{k_j} + \beta_j,$$

where $p_j \in \{2, 3, 5\}$ and $\alpha_j, \beta_j \in \langle \omega_1, \dots, \omega_n \rangle$. \square

Example 2.7. For the cases $G = SU(n), Sp(n), E_6$ and E_7 we refer to Theorem 5.1 for the explicit form of the formula (2.13) of the rings $H^*(G/T)$. \square

3 Construction and computation in $E_3^{*,*}(G)$

By the fiber degrees the third page of the spectral sequence $\{E_r^{*,*}(G), d_r\}$ has the decomposition $E_3^{*,*}(G) = E_3^{*,0}(G) \oplus E_3^{*,1}(G) \oplus \dots \oplus E_3^{*,N}(G)$, $N = \dim T$. Based on Theorems 2.4 and 2.6 we single out certain elements in the initial two summands $E_3^{*,0}$ and $E_3^{*,1}$, which will show to generate the ring $H^*(G)$ multiplicatively. Along the way we demonstrate that, with respect to our explicit constructions of the generators on the ring $H^*(G)$, the Bockstein homomorphism, Steenrod operators, as well as the homomorphism θ in (1.8), can be effectively evaluated by simple formulae, see Section 3.4 and Theorem 3.10.

3.1 The term $E_3^{*,0}(G)$

The formula (2.3) of the differential $d_2 : E_2^{*,*}(G) \rightarrow E_2^{*,*}(G)$ implies that $E_3^{*,0}(G) = H^*(G/T) / \langle \text{Im } \tau \rangle$. From (2.13) one gets

Lemma 3.1. $E_3^{*,0}(G) = H^*(G/T) |_{\tau(t_1) = \dots = \tau(t_N) = 0}$. \square

3.2 Constructions in $E_3^{*,1}(G)$

For a d_2 -cocycle $\gamma \in E_2^{*,*}(G)$ write $[\gamma] \in E_3^{*,*}(G)$ for its cohomology class. Based on Theorems 2.4 and 2.6, we present two ways to construct elements in $E_3^{*,1}(G)$. The first one resorts to $\ker \tau$, while the second utilizes $\text{Im } \tau$.

For a $t \in \ker \tau$ the element $1 \otimes t \in E_2^{0,1}$ is clear a d_2 -cocycle. The class $\iota(t) := [1 \otimes t] \in E_3^{0,1}(G)$ will be called a *primary 1-form* of G with base degree 0. Moreover, in view of the exact sequence (2.5) one has

Lemma 3.2. *The map $\iota : \ker \tau \rightarrow E_3^{0,1}(G) = H^1(G)$ is an isomorphism.* \square

For the second construction we take firstly the ring \mathbb{Z} of integers as coefficient for cohomologies. In view of the formula (2.13) one has the surjective ring map

$$f : \mathbb{Z}[\omega_i, y_j]_{1 \leq i \leq n, 1 \leq j \leq m} \rightarrow H^*(G/T) \text{ with } \ker f = \langle h_i, f_j, g_j \rangle.$$

Since f is an isomorphism in degree 2 the transgression τ has a unique lift $\tilde{\tau}$ into the free polynomial ring $\mathbb{Z}[\omega_i, y_j]$ subject to the relation $\tau = f \circ \tilde{\tau}$. For a polynomial $P \in \langle \text{Im } \tilde{\tau} \rangle$ we can write

(3.1) $P = p_1 \cdot \tilde{\tau}(t_1) + \cdots + p_N \cdot \tilde{\tau}(t_N)$ with $p_i \in \mathbb{Z}[\omega_i, y_j]$,

and set $\tilde{P} := f(p_1) \otimes t_1 + \cdots + f(p_N) \otimes t_N \in E_2^{*,1}(G)$. It is crucial for us to note from $f(P) = d_2(\tilde{P})$ (by (2.3)) that

$$\text{"} P \in \langle \text{Im } \tilde{\tau} \rangle \cap \ker f \text{ implies } \tilde{P} \in \ker d_2 \text{"}.$$

Most importantly, one can show that

Lemma 3.3. *The map $\varphi : \langle \text{Im } \tilde{\tau} \rangle \cap \ker f \rightarrow E_3^{*,1}(G)$ by $\varphi(P) = [\tilde{P}]$ is well defined.*

Proof. We are bound to show the class $[\tilde{P}] \in E_3^{*,1}(G)$ is independent of a choice of the expansion (3.1). Assume in addition to (3.1) that one has a second expansion $P = h_1 \cdot \tilde{\tau}(t_1) + \cdots + h_N \cdot \tilde{\tau}(t_N)$. Then the equation

$$(p_1 - h_1) \cdot \tilde{\tau}(t_1) + \cdots + (p_N - h_N) \cdot \tilde{\tau}(t_N) = 0$$

holds in the ring $\mathbb{Z}[\omega_i, y_j]$. We can assume below that $p_1 - h_1 \neq 0$.

Case 1. The set $\{\tilde{\tau}(t_1), \dots, \tilde{\tau}(t_N)\}$ is a basis of $\text{Im } \tilde{\tau}$: Since $\{\tilde{\tau}(t_1), \dots, \tilde{\tau}(t_N)\} \subset \mathbb{Z}[\omega_i, y_j]$ is algebraically independent, the above equation implies that all the differences $p_i - h_i$ with $i \neq 1$ are divisible by $\tilde{\tau}(t_1)$. That is $p_i - h_i = q_i \cdot \tilde{\tau}(t_1)$ for some $q_i \in \mathbb{Z}[\omega_i, y_j]$, $2 \leq i \leq N$. The proof is done by the calculation

$$\begin{aligned} & d_2(f(q_2) \otimes t_1 t_2 + \cdots + f(q_N) \otimes t_1 t_N) \\ &= f(p_1 - h_1) \otimes t_1 + \cdots + f(p_N - h_N) \otimes t_N. \end{aligned}$$

Case 2. The set $\{\tilde{\tau}(t_1), \dots, \tilde{\tau}(t_N)\}$ is linearly dependent $\text{Im } \tilde{\tau}$: Take a subset $\{\bar{t}_1, \dots, \bar{t}_{n'}\} \subset H^1(T)$ ($n' \leq N$) so that its $\tilde{\tau}$ -image is a basis of the group $\text{Im } \tilde{\tau}$. Let $B = (b_{ij})_{n' \times N}$ be the matrix expressing the elements $\tau(t_i)$ by the basis elements $\{\tilde{\tau}(\bar{t}_1), \dots, \tilde{\tau}(\bar{t}_{n'})\}$. Denote by B^τ the transpose of B and set

$$(p'_1, \dots, p'_{n'}) = (p_1, \dots, p_N) \cdot B^\tau, (h'_1, \dots, h'_{n'}) = (h_1, \dots, h_N) \cdot B^\tau.$$

Then, in addition to the next two expansions of P in $\mathbb{Z}[\omega_i, y_j]_{1 \leq i \leq n, 1 \leq j \leq m}$

$$P = p'_1 \cdot \tilde{\tau}(\bar{t}_1) + \cdots + p'_{n'} \cdot \tilde{\tau}(\bar{t}_{n'}) = h'_1 \cdot \tilde{\tau}(\bar{t}_1) + \cdots + h'_{n'} \cdot \tilde{\tau}(\bar{t}_{n'}),$$

one has the following relations in $E_2^{*,1}(G)$

$$\begin{aligned} & f(p_1) \otimes t_1 + \cdots + f(p_N) \otimes t_N = f(p'_1) \otimes \bar{t}_1 + \cdots + f(p'_{n'}) \otimes \bar{t}_{n'}, \\ & f(h_1) \otimes t_1 + \cdots + f(h_N) \otimes t_N = f(h'_1) \otimes \bar{t}_1 + \cdots + f(h'_{n'}) \otimes \bar{t}_{n'}. \end{aligned}$$

The proof is done by the following computation in $E_3^{*,1}(G)$

$$\begin{aligned} & [f(p_1) \otimes t_1 + \cdots + f(p_N) \otimes t_N] = [f(p'_1) \otimes \bar{t}_1 + \cdots + f(p'_{n'}) \otimes \bar{t}_{n'}] \\ &= [f(h'_1) \otimes \bar{t}_1 + \cdots + f(h'_{n'}) \otimes \bar{t}_{n'}] = [f(h_1) \otimes t_1 + \cdots + f(h_N) \otimes t_N], \end{aligned}$$

where the second equality has been shown in Case 1. \square

The maps $\tilde{\tau}$ and φ above has its analogue for cohomology over a finite field \mathbb{F}_p . Precisely, from Theorem 2.6 one can deduce a presentation of the ring $H^*(G/T; \mathbb{F}_p)$ in the following form ([8, Lemma 2.3])

$$(3.2) \quad H^*(G/T; \mathbb{F}_p) = \mathbb{F}_p[\omega_1, \dots, \omega_n, y_t] / \left\langle \delta_1, \dots, \delta_n, y_t^{k_t} + \sigma_t \right\rangle_{t \in E(G, p)},$$

where $\delta_s \in \mathbb{F}_p[\omega_1, \dots, \omega_n]$, $\sigma_t \in \langle \omega_1, \dots, \omega_n \rangle$, $E(G, p) = \{1 \leq t \leq m; p_t = p\}$. Based on (3.2) one formulates the \mathbb{F}_p -analogies of the maps f , $\tilde{\tau}$ and φ as

$$\begin{aligned} f_p : \mathbb{F}_p[\omega_1, \dots, \omega_n, y_t]_{t \in E(G, p)} &\rightarrow H^*(G/T; \mathbb{F}_p), \\ \tilde{\tau}_p : H^1(T; \mathbb{F}_p) &\rightarrow \mathbb{F}_p[\omega_1, \dots, \omega_n, y_t]_{t \in E(G, p)}, \\ \varphi_p : \langle \text{Im } \tilde{\tau}_p \rangle \cap \langle \ker f_p \rangle &\rightarrow E_3^{*,1}(G; \mathbb{F}_p). \end{aligned}$$

The proof of Lemma 3.3 is applicable to show that

Lemma 3.4. *The correspondence φ_p is well defined. In particular*

$$\varphi_p(P'P) = 0 \text{ if } P' \in \langle \text{Im } \tilde{\tau}_p \rangle \text{ and } P \in \ker f_p. \square$$

3.3 Extension from $E_3^{*,*}(G)$ to $H^*(G)$

In preparation to solve the extension problem from $E_3^{*,*}(G)$ to $H^*(G)$ we let F^p be the filtration on $H^*(G)$ defined by the fibration (2.1). That is

$$0 = F^{r+1}(H^r(G)) \subseteq F^r(H^r(G)) \subseteq \dots \subseteq F^0(H^r(G)) = H^r(G)$$

with $E_\infty^{p,q}(G) = F^p(H^{p+q}(G))/F^{p+1}(H^{p+q}(G))$. The routine relation $d_r(E_r^{*,0}(G)) = 0$ for $r \geq 2$ yields the sequence of quotient maps

$$H^r(G/T) = E_2^{r,0} \rightarrow E_3^{r,0} \rightarrow \dots \rightarrow E_\infty^{r,0} = F^r(H^r(G)) \subset H^r(G)$$

whose composition agrees with the induces map $\pi^* : H^*(G/T) \rightarrow H^*(G)$ [15, P.147]. For this reason we can reserve π^* also for the composition

$$(3.3) \quad \pi^* : E_3^{*,0}(G) \rightarrow \dots \rightarrow E_\infty^{*,0}(G) = F^r(H^r(G)) \subset H^r(G).$$

The property $H^{odd}(G/T) = 0$ by Theorem 3.1 indicates that $E_r^{odd,q} = 0$ for $r, q \geq 0$. This implies that $F^{2k+1}(H^{2k+1}(G)) = F^{2k+2}(H^{2k+1}(G)) = 0$ and that

$$E_\infty^{2k,1}(G) = F^{2k}(H^{2k+1}(G)) \subset H^{2k+1}(G).$$

Combining this with $d_r(E_r^{*,1}) = 0$ for $r \geq 3$ yields the composition

$$(3.4) \quad \kappa : E_3^{*,1}(G) \rightarrow \dots \rightarrow E_\infty^{*,1}(G) \subset H^{2t+1}(G)$$

that interprets directly elements of $E_3^{*,1}$ as cohomology classes of the group G .

Definition 3.5. For a polynomial $P \in \ker f \cap \langle \text{Im } \tilde{\tau} \rangle$ (resp. $P \in \ker f_p \cap \langle \text{Im } \tilde{\tau}_p \rangle$) we shall refer to the class $\kappa\varphi(P) \in H^*(G)$ (resp. $\kappa\varphi_p(P) \in H^*(G; \mathbb{F}_p)$) as *the primary 1-form on G with characteristic polynomial P* .

Example 3.6. If the group G is 1-connected then $\langle \text{Im } \tilde{\tau} \rangle = \langle \omega_1, \dots, \omega_n \rangle$ (resp. $\langle \text{Im } \tilde{\tau}_p \rangle = \langle \omega_1, \dots, \omega_n \rangle$) by Theorem 2.4, and the presentation (2.13) implies that the set of polynomials

$$S(G) = \{h_i, p_j \beta_j - y_j^{k_j} \alpha_j \mid 1 \leq i \leq k, 1 \leq j \leq m\}$$

(resp. $S_p(G) := \{\delta_1, \dots, \delta_n\}$)

belongs to $\ker f \cap \langle \text{Im } \tilde{\tau} \rangle$ (resp. to $\ker f_p \cap \langle \text{Im } \tilde{\tau}_p \rangle$). It will be called a *set of primary characteristic polynomials* of G over \mathbb{Z} (resp. over \mathbb{F}_p).

It has been shown in [7, 8] that

- i) the square free products of the primary 1-forms $\kappa \circ \varphi(P)$ with $P \in S(G)$ is a basis of the free part of the integral cohomology $H^*(G)$;
- ii) the ring $H^*(G; \mathbb{F}_p)$ is generated by the set $\{\kappa \circ \varphi_p(\delta) \mid \delta \in S_p(G)\}$ of primary 1-forms, together with $\pi^*(y_t)$, $t \in E(G, p)$.

In Section 4 these results will act as the input for computing the cohomologies of the adjoint Lie groups PG . \square

3.4 Steenrod operations on $H^*(G; \mathbb{F}_p)$

Let \mathcal{A}_p be the Steenrod algebra with $\mathcal{P}^k \in \mathcal{A}_p$, $k \geq 1$, the k^{th} reduced power (if $p = 2$ it is also customary to write Sq^{2k} instead of \mathcal{P}^k), and with $\delta_p = r_p \circ \beta_p \in \mathcal{A}_p$ the Bockstein operator [22]. We can reduce the \mathcal{A}_p action on the primary 1-forms $\kappa \varphi_p(P)$ to computation with the characteristic polynomials P .

Since the Koszul complex $E_2^{*,*}(G)$ is torsion free one has for a prime p the short exact sequence of complexes

$$0 \rightarrow E_2^{*,*}(G) \xrightarrow{\cdot p} E_2^{*,*}(G) \xrightarrow{r_p} E_2^{*,*}(G; \mathbb{F}_p) \rightarrow 0.$$

With respect to the maps κ and π^* the connecting homomorphism $\widehat{\beta}_p$ of the associated cohomology exact sequence clearly satisfies the commutative diagram

$$(3.5) \quad \begin{array}{ccc} E_3^{*,1}(G; \mathbb{F}_p) & \xrightarrow{\widehat{\beta}_p} & E_3^{*,0}(G) \\ \kappa \downarrow & & \pi^* \downarrow \\ H^*(G; \mathbb{F}_p) & \xrightarrow{\beta_p} & H^*(G) \end{array}.$$

In addition, by Lemma 3.1 the quotient map $H^*(G/T) \rightarrow E_3^{*,0}(G)$ is

$$f(P) \rightarrow f(P) \mid_{\tau(t_1) = \dots = \tau(t_N) = 0}, P \in \mathbb{Z}[\omega_i, y_j].$$

Let $P_0 \in \langle \text{Im } \tilde{\tau} \rangle$ be an integral lift of a polynomial $P \in \ker f_p \cap \langle \text{Im } \tilde{\tau}_p \rangle$ (i.e. $P_0 \equiv P \pmod{p}$). The diagram chasing

$$\begin{array}{ccc} \varphi(P_0) & \xrightarrow{r_p} & \varphi_p(P) \\ d_2 \downarrow & & d_2 \downarrow \\ \frac{1}{p} f(P_0) & \xrightarrow{\cdot p} & f(P_0) & 0 \end{array}$$

in above short exact sequence shows that

Lemma 3.7. $\beta_p(\kappa \circ \varphi_p(P)) = \pi^*(\frac{1}{p} f(P_0) \mid_{\tau(t_1) = \dots = \tau(t_N) = 0})$. \square

Let $c : (G_0, T_0) \rightarrow (G, T)$ be the universal covering of a semi-simple Lie group G , and consider the fibration induced by the inclusion $i : T_0 \rightarrow G_0$

$$(3.6) \quad G/T \xrightarrow{\psi} BT_0 \xrightarrow{Bi} BG_0,$$

where BT_0 (resp. BG_0) is the classifying space of the group T_0 (resp. G_0). It is known that the compositions $\tilde{s}_\alpha := \psi \circ s_\alpha : S^2 \rightarrow BT_0$ with $\alpha \in \Delta$ represent a basis of the group $H_2(BT_0)$, where s_α are the maps defined by (2.6). As a result we can also write $\{\omega_1, \dots, \omega_n\}$ for the basis on $H^2(BT_0)$ Kronnecker dual to the ordered basis $\{\tilde{s}_\alpha \mid \alpha_i \in \Delta\}$ on $H_2(BT_0)$. In this sense $\psi^* = \bar{f}_p$, where \bar{f}_p denotes the restriction of f_p on the subalgebra $H^*(BT_0; \mathbb{F}_p) = \mathbb{F}_p[\omega_1, \dots, \omega_n]$. Since the lift $\tilde{\tau}_p$ of the transgression τ takes values in $H^*(BT_0; \mathbb{F}_p)$ one has the subspace

$$\ker \bar{f}_p \cap \langle \text{Im } \tilde{\tau}_p \rangle \subset H^*(BT_0; \mathbb{F}_p)$$

which is clearly closed under the \mathcal{A}_p action on $H^*(BT_0; \mathbb{F}_p)$. The proof of [8, Lemma 3.2] is applicable to show the following formula, that reduces the \mathcal{P}^k action on $H^*(G; \mathbb{F}_p)$ to that on the much simpler \mathcal{A}_p -algebra $H^*(BT_0; \mathbb{F}_p)$.

Lemma 3.8. *For a characteristic polynomial $P \in \ker \bar{f}_p \cap \langle \text{Im } \tilde{\tau}_p \rangle$ one has*

$$(3.7) \quad \mathcal{P}^k(\kappa \circ \varphi_p(P)) = \kappa \circ \varphi_p(\mathcal{P}^k(P)). \square$$

3.5 A refinement of the exact sequence (1.8)

Returning to the situation concerned by Theorem 1.3 let G be a semi-simple Lie group G whose center $\mathcal{Z}(G)$ contains the cyclic group \mathbb{Z}_q . Then the circle bundle C on the quotient group G/\mathbb{Z}_q fits into the commutative diagram

$$(3.8) \quad \begin{array}{ccccccc} S^1 & \hookrightarrow & [T \times S^1] / \mathbb{Z}_q & \xrightarrow{C'} & T' \\ \parallel & & \cap & & \cap \\ S^1 & \hookrightarrow & [G \times S^1] / \mathbb{Z}_q & \xrightarrow{C} & G / \mathbb{Z}_q, \\ & & \pi' \downarrow & & \pi \downarrow \\ & & G / T & = & G / T \end{array}$$

where $T \subset G$ is a fixed maximal torus on G , $T' := T / \mathbb{Z}_q$, the vertical maps π' and π are the obvious quotients by the maximal torus, and where C' denotes the restriction of C to $[T \times S^1] / \mathbb{Z}_q$. Since the maximal torus $[T \times S^1] / \mathbb{Z}_q$ of $[G \times S^1] / \mathbb{Z}_q$ has the factorization $T' \times S^1$ so that C' is the projection onto the first factor (by the proof of Theorem 1.2), one can take a basis $\Theta = \{\theta_1, \dots, \theta_n, \theta_0\}$ ($n = \dim T'$) for the unit lattice of the group $[G \times S^1] / \mathbb{Z}_q$, so that the tangent map of C' at the group unit e carries the subset $\{\theta_1, \dots, \theta_n\}$ to a basis of the unit lattice of G / \mathbb{Z}_q . As a result if we let $\{t_1, \dots, t_n, t_0\}$ be the basis of $H^1([T \times S^1] / \mathbb{Z}_q)$ corresponding to Θ in the manner of (2.8), then

- a) C'^* maps $H^*(T')$ isomorphically onto the subring $\Lambda^*(t_1, \dots, t_n)$ of $H^*([T \times S^1] / \mathbb{Z}_q) = \Lambda^*(t_1, \dots, t_n, t_0)$;
- b) the transgression τ in π is the restriction of τ' on $H^1(T')$.

Summarizing, the bundle map C from π' to π fits into the short exact sequence

$$(3.9) \quad 0 \rightarrow E_2^{*,k}(G / \mathbb{Z}_q) \xrightarrow{C^*} E_2^{*,k}([G \times S^1] / \mathbb{Z}_q) \xrightarrow{\bar{\theta}} E_2^{*,k-1}(G / \mathbb{Z}_q) \rightarrow 0,$$

where the quotient map $\bar{\theta}$ has the following description: if $x \in H^*(G / T)$, $y = y_0 + t_0 \cdot y_1 \in H^*([T \times S^1] / \mathbb{Z}_q)$ with $y_0, y_1 \in \Lambda^*(t_1, \dots, t_n)$, then

$$(3.10) \quad \overline{\theta}(x \otimes y) = x \otimes y_1.$$

It follows from (3.9) that one has an exact sequence of the form

$$(3.11) \quad \cdots \rightarrow E_3^{*,r}(G/Z_q) \xrightarrow{C^*} E_3^{*,r}([G \times S^1]/Z_q) \xrightarrow{\overline{\theta}} E_3^{*,r-1}(G/Z_q) \xrightarrow{\varpi} E_3^{*,r-1}(G/Z_q) \xrightarrow{C^*} \cdots$$

in which the map ϖ is induced by the endomorphism $x \otimes y \rightarrow (x \cup \varpi) \otimes y$ on $E_2^{*,*}(G/\mathbb{Z}_q)$ with $\varpi := \tau'(t_0)$.

With respect to the multiplicative structure inherited from that on $E_2^{*,*}$ the third page $E_3^{*,*}$ is a bi-graded ring [?, P.668]. Let $J(\varpi)$ and $\langle \varpi \rangle$ be respectively the subring and the ideal of $E_3^{*,*}(G/\mathbb{Z}_q)$ generated by the class $\varpi \in E_3^{2,0}(G/\mathbb{Z}_q)$. Write $E_3^{*,*}(G/\mathbb{Z}_q)_{\langle \varpi \rangle}$ for the quotient ring $E_3^{*,*}(G/\mathbb{Z}_q)/\langle \varpi \rangle$ with quotient map g . Then, in addition to the exact sequence

$$(3.12) \quad 0 \rightarrow \langle \varpi \rangle \rightarrow E_3^{*,*}(G/\mathbb{Z}_q) \xrightarrow{g} E_3^{*,*}(G/\mathbb{Z}_q)_{\langle \varpi \rangle} \rightarrow 0$$

the proof of Theorem 1.3 is valid to show the following result. By Lemma 3.2 the group $E_3^{0,1}([G \times S^1]/\mathbb{Z}_q)$ has an element that corresponds to the class ξ_1 defined by (1.6), which we denote still by ξ_1 .

Theorem 3.9. *The induced map C^* fits in the exact sequence*

$$(3.13) \quad 0 \rightarrow E_3^{*,*}(G/\mathbb{Z}_q)_{\langle \varpi \rangle} \xrightarrow{C^*} E_3^{*,*}([G \times S^1]/\mathbb{Z}_q) \xrightarrow{\overline{\theta}} E_3^{*,*}(G/\mathbb{Z}_q) \xrightarrow{\varpi} \langle \varpi \rangle \rightarrow 0$$

where

- i) $\overline{\theta}(\xi_1) = q \in E_3^{0,0}(G/\mathbb{Z}_q)$;
- ii) $\overline{\theta}(x \cup C^*(y)) = \overline{\theta}(x) \cup y$, $x \in E_3^{*,*}([G \times S^1]/\mathbb{Z}_q)$, $y \in E_3^{*,*}(G/\mathbb{Z}_q)$,
- iii) the class ϖ satisfies $\pi^*(\varpi) = \omega$ (see (3.3)).

In addition, if the map g admits a split homomorphism $j : E_3^{*,*}(G/\mathbb{Z}_q)_{\langle \varpi \rangle} \rightarrow E_3^{*,*}(G/\mathbb{Z}_q)$, then the map

$$h : J(\varpi) \otimes E_3^{*,*}(G/\mathbb{Z}_q)_{\langle \varpi \rangle} \rightarrow E_3^{*,*}(G/\mathbb{Z}_q)$$

by $h(\varpi^r \otimes x) = \varpi^r \cup j(x)$ induces an isomorphism of $J(\varpi)$ -modules

$$(3.14) \quad E_3^{*,*}(G/\mathbb{Z}_q) \cong \frac{J(\varpi) \otimes E_3^{*,*}(G/\mathbb{Z}_q)_{\langle \varpi \rangle}}{\langle \varpi \cdot \text{Im } \overline{\theta} \rangle}. \square$$

Proof. It suffices to show the relations i), ii) and iii). By the choice of the class $\xi_1 \in E_3^{0,1}([G \times S^1]/\mathbb{Z}_q)$ and by the following commutative diagram

$$\begin{array}{ccc} E_3^{0,1}([G \times S^1]/\mathbb{Z}_q) & \xrightarrow{\overline{\theta}} & E_3^{0,0}(G/\mathbb{Z}_q) \\ \parallel & & \parallel \\ H^1([G \times S^1]/\mathbb{Z}_q) & \xrightarrow{\theta} & H^0(G/\mathbb{Z}_q) \end{array},$$

property i) corresponds to i) of Theorem 1.3, while ii) comes directly from the formula (3.10). Finally, the relation iii) is shown by the commutative diagram induced by the map π^* in (3.3):

$$\begin{array}{ccc} E_3^{0,0}(G/\mathbb{Z}_q) & \xrightarrow{\cup \varpi} & E_3^{2,0}(G/\mathbb{Z}_q) \\ \parallel & \pi^* \downarrow & \\ H^0(G/\mathbb{Z}_q) & \xrightarrow{\cup \omega} & H^2(G/\mathbb{Z}_q) \end{array}. \square$$

The exact sequence (3.13) can be seen to be a refinement of the sequence (1.8). In addition, the maps π^* and κ in (3.3) and (3.4) build up the obvious commutative diagram relating the operator θ in (1.8) with the map $\bar{\theta}$ in (3.13)

$$(3.15) \quad \begin{array}{ccc} E_3^{2r,1}([G \times S^1] / \mathbb{Z}_q) & \xrightarrow{\bar{\theta}} & E_3^{2r,0}(G / \mathbb{Z}_q) \\ \kappa \downarrow & & \pi^* \downarrow \\ H^{2r+1}(G \times S^1) & \xrightarrow{\theta} & H^{2r}(G / \mathbb{Z}_q) \end{array} .$$

Moreover, the map $\bar{\theta}$ (hence θ) admits a simple formula we come to describe.

For a polynomial $P \in \langle \text{Im } \tilde{\tau}' \rangle$ we set $P_0 = P|_{\tilde{\tau}(t_1)=\dots=\tilde{\tau}(t_n)=0}$. Then $P - P_0 \in \langle \text{Im } \tilde{\tau}' \rangle$ and P_0 is divisible by $\varpi = \tilde{\tau}'(t_0)$. This enables us to define the derivation $\partial P / \partial \varpi$ of P with respect to ϖ by the formula

$$(3.16) \quad \partial P / \partial \varpi := P_0 / \varpi,$$

and to get an expansion of P in the form (note that $\tilde{\tau}'(t_i) = \tilde{\tau}(t_i)$, $i \leq n$, by b))

$$P = p_1 \cdot \tilde{\tau}'(t_1) + \dots + p_n \cdot \tilde{\tau}'(t_n) + \partial P / \partial \varpi \cdot \tilde{\tau}'(t_0).$$

In term of Lemma 3.3 for a polynomial $P \in \ker f \cap \langle \text{Im } \tilde{\tau}' \rangle$ we have

$$\varphi(P) = [f(p_1) \otimes t_1 + \dots + f(p_n) \otimes t_n + f(\partial P / \partial \varpi) \otimes t_0].$$

The diagram (3.15), together with the formula (3.10), concludes that

Theorem 3.10. *The map $\bar{\theta}$ in (3.13) (resp. the map θ in (1.8)) satisfies that*

$$(3.17) \quad \bar{\theta}(\varphi(P)) = f(\partial P / \partial \varpi) \text{ (resp. } \theta(\kappa \circ \varphi(P)) = \pi^* f(\partial P / \partial \varpi)),$$

where $P \in \ker f \cap \langle \text{Im } \tilde{\tau}' \rangle$, $P_0 = P|_{\tilde{\tau}(t_1)=\dots=\tilde{\tau}(t_n)=0}$. \square

Let $P \in \ker f \cap \langle \text{Im } \tilde{\tau}' \rangle$ be a polynomial with $\bar{\theta}(\varphi(P)) = 0$. By the exact sequence (3.13) there exists a 1-form $\eta \in E_3^{*,1}(G / \mathbb{Z}_q)$ satisfying $C^*(\eta) = \varphi(P)$. The proof of the next result indicates an algorithm to construct from P a characteristic polynomial $P' \in \ker f \cap \langle \text{Im } \tilde{\tau}' \rangle$ for such a class η .

Lemma 3.11. *For a polynomial $P \in \ker f \cap \langle \text{Im } \tilde{\tau}' \rangle$ with $\bar{\theta}(\varphi(P)) = 0$, there exists a polynomial $P' \in \ker f \cap \langle \text{Im } \tilde{\tau}' \rangle$ such that $C^*\varphi(P') = \varphi(P)$.*

Proof. With $\bar{\theta}(\varphi(P)) = 0$ we have by the exact sequence (3.13) that $d_2(\gamma) = f(\partial P / \partial \varpi)$ for some $\gamma \in E_2^{*,1}(G / \mathbb{Z}_q)$. We can assume further that $\gamma = \tilde{H}$ for some $H = h_1 \cdot \tilde{\tau}(t_1) + \dots + h_n \cdot \tilde{\tau}(t_n) \in \langle \text{Im } \tilde{\tau}' \rangle$. The desired polynomial P' is given by $P' := P + (H - \partial P / \partial \varpi) \cdot \varpi$.

Indeed, the obvious relation $H - \partial P / \partial \varpi \in \ker f$ implies that

$$(H - \partial P / \partial \varpi) \cdot \varpi \in \langle \text{Im } \tilde{\tau}' \rangle \cap \ker f.$$

Consequently, $P' \in \langle \text{Im } \tilde{\tau}' \rangle \cap \ker f$. The relation $C^*(\varphi(P')) = \varphi(P)$ on $E_3^{*,1}([G \times S^1] / \mathbb{Z}_q)$ is verified by the following calculation in $E_2^{*,1}([G \times S^1] / \mathbb{Z}_q)$

$$\begin{aligned} d_2\left(\sum_{1 \leq i \leq n} f(h_i) \otimes t_i t_0\right) &= -f(\partial P / \partial \varpi) \otimes t_0 + \sum_{1 \leq i \leq n} f(h_i) \varpi \otimes t_i \\ &= \tilde{P}' - \tilde{P}. \square \end{aligned}$$

4 The cohomology of adjoint Lie groups

A Lie group G is called *adjoint* if its center subgroup $\mathcal{Z}(G)$ is trivial. In particular, for any Lie group G the quotient group $PG := G/\mathcal{Z}(G)$ is adjoint. Granted with the constructions and formulae developed in Section 3 we compute in this section the cohomologies of the adjoint Lie groups PG for $G = SU(n), Sp(n), E_6, E_7$. In these cases the quotient maps $c : G \rightarrow PG$ are cyclic. The corresponding circle bundle over PG is denoted by

$$C : [G \times S^1] / \mathbb{Z}_q \cong G \times S^1 \rightarrow PG, \quad q = |\mathcal{Z}(G)|,$$

where the diffeomorphism \cong follows from Theorem 1.2.

Briefly, the computation is carried out by three steps. Starting from the formula (2.13) of the ring $H^*(G/T)$ we obtain firstly the subgroups $\text{Im } \pi^*$ and $\text{Im } \kappa$. The exact sequence (3.13) is then applied to formulate the additive cohomology $H^*(PG)$ by $\text{Im } \pi^*$ and $\text{Im } \kappa$. Finally, the structure of $H^*(PG)$ as a ring is determined by expressing the squares x^2 with $x \in \text{Im } \kappa$ as elements of $\text{Im } \pi^*$, see [7, Lemma 2.8].

With the group G being 1-connected the cohomologies $H^*(G; \mathbb{F}_p)$ are known [7]. The following result allows us to exclude the cohomology $H^*(PG; \mathbb{F}_p)$ with $(p, q) = 1$ from further consideration.

Theorem 4.1. *If $(p, q) = 1$ the map C induces a ring isomorphism*

$$C^* : H^*(PG; \mathbb{F}_p) \cong H^*(G; \mathbb{F}_p).$$

Proof. With p co-prime to q one has $\omega \equiv 0 \pmod{p}$. Therefore the exact sequence (1.7) becomes

$$0 \rightarrow H^*(PG; \mathbb{F}_p) \xrightarrow{C^*} H^*(G \times S^1; \mathbb{F}_p) \xrightarrow{\theta} H^*(PG; \mathbb{F}_p) \rightarrow 0.$$

Moreover, the relation $\theta(\xi_1) = 1$ by i) of Theorem 1.3 implies that, with respect to the decomposition $H^*(G; \mathbb{F}_p) \oplus \xi_1 \cdot H^*(G; \mathbb{F}_p)$ on $H^*(G \times S^1; \mathbb{F}_p)$ the map C^* carries $H^*(PG; \mathbb{F}_p)$ injectively into the first summand $H^*(G; \mathbb{F}_p)$, while by ii) of Theorem 1.3 the map θ maps the second summand $\xi_1 \cdot H^*(G; \mathbb{F}_p)$ surjectively onto $H^*(PG; \mathbb{F}_p)$. This establishes the isomorphism. \square

4.1 The map $C^* : E_3^{*,0}(G/\mathbb{Z}_q) \rightarrow E_3^{*,0}(G \times S^1)$ in (3.11)

We begin by taking integers as coefficients for cohomology. Since the groups G are 1-connected $\text{Tor}H^2(G \times S^1) = 0$ by Theorem 1.2. Therefore, the transgression τ' in π' (see (3.8)) is surjective by (2.2). One gets from Theorem 5.1, as well the formula $E_3^{*,0}(G \times S^1) = H^*(G/T) |_{\omega_1 = \dots = \omega_n = 0}$ by Lemma 3.1, that

$$(4.1) \quad \begin{aligned} E_3^{*,0}(SU(n) \times S^1) &= E_3^{*,0}(Sp(n) \times S^1) = \mathbb{Z}; \\ E_3^{*,0}(E_6 \times S^1) &= \frac{\mathbb{Z}[x_6, x_8]}{\langle 2x_6, 3x_8, x_6^2, x_8^3 \rangle}; \\ E_3^{*,0}(E_7 \times S^1) &= \frac{\mathbb{Z}[x_6, x_8, x_{10}, x_{18}]}{\langle 2x_6, 3x_8, 2x_{10}, 2x_{18}, x_6^2, x_8^3, x_{10}^2, x_{18}^2 \rangle}. \end{aligned}$$

where x_i 's are the special Schubert classes on E_n/T , $n = 6, 7$, defined by (5.1).

Similarly, for the groups PG take a set $\Omega = \{\phi_1, \dots, \phi_m\}$ of fundamental dominant weights as a basis for the unit lattice Λ_e of PG , and let $\{t_1, \dots, t_m\}$ be

the corresponding basis on the group $H^1(T')$ (see in (3.8)), where $m = n-1, n, 6$ or 7 in accordance to $G = SU(n), Sp(n), E_6, E_7$. By iii) of Lemma 2.2 the matrix $C(\Omega)$ expressing the basis Δ of the root lattice Λ_r by Ω is the Cartan matrix of G . Granted with the presentation of the rings $H^*(G/T)$ in Theorem 5.1, as well as the results of Lemma 5.2, the formula $E_3^{*,0}(PG) = H^*(G/T) |_{\tau(t_1)=\dots=\tau(t_m)=0}$ by Lemma 3.1 yields that

$$(4.2) \quad \begin{aligned} E_3^{*,0}(PSU(n)) &= \frac{\mathbb{Z}[\omega_1]}{\langle b_{n,r}\omega_1^r, 1 \leq r \leq n \rangle} \text{ with } b_{n,r} = \text{g.c.d.}\{ \binom{n}{1}, \dots, \binom{n}{r} \}; \\ E_3^{*,0}(PSp(n)) &= \frac{\mathbb{Z}[\omega_1]}{\langle 2\omega_1, \omega_1^{2r+1} \rangle}, n = 2^r(2s+1); \\ E_3^{*,0}(PE_6) &= \frac{\mathbb{Z}[\omega_1, x_3', x_4]}{\langle 3\omega_1, 2x_3', 3x_4, x_3'^2, \omega_1^9, x_4^3 \rangle}, x_3' = x_3 + \omega_1^3; \\ E_3^{*,0}(PE_7) &= \frac{\mathbb{Z}[\omega_2, x_3, x_4, x_6, x_9]}{\langle 2\omega_2, \omega_2^2, 2x_3, 3x_4, 2x_5, 2x_9, x_3^2, x_4^2, x_5^2, x_9^2 \rangle}, \end{aligned}$$

where $\binom{n}{r} := \frac{n!}{r!(n-r)!}$. Inputting (4.1) and (4.2) into the section

$$E_3^{*,0}(PG) \xrightarrow{\cong} E_3^{*,0}(PG) \xrightarrow{C^*} E_3^{*,0}(G \times S^1) \rightarrow 0$$

of the exact sequence (3.11) one obtains that

Lemma 4.2. *In the order of $G = SU(n), Sp(n), E_6$ and E_7 one has*

- i) $J(\varpi) = \frac{\mathbb{Z}[\varpi]}{\langle b_{n,r}\varpi^r, 1 \leq r \leq n \rangle}, \frac{\mathbb{Z}[\varpi]}{\langle 2\varpi, \varpi^{2r+1} \rangle}, \frac{\mathbb{Z}[\varpi]}{\langle 3\varpi, \varpi^9 \rangle}, \frac{\mathbb{Z}[\varpi]}{\langle 2\varpi, \varpi^2 \rangle};$
- ii) the map $C^* : E_3^{*,0}(PG) \rightarrow E_3^{*,0}(G \times S^1)$ is given by
 $C^*(x_s) = x_s, C^*(x_3') = x_3; C^*(\varpi) = 0.$

where $\varpi = \omega_1$ for $G = SU(n), Sp(n), E_6$; $\varpi = \omega_2$ for $G = E_7$. \square

Since $H^*(G/T)$ is torsion free $E_3^{*,0}(X; \mathbb{F}_p) = E_3^{*,0}(X) \otimes \mathbb{F}_p$ for both $X = G/\mathbb{Z}_q$ and $G \times S^1$. Let $J_p(\varpi) \subset E_3^{*,0}(PG; \mathbb{F}_p)$ be the subring generated by ϖ and set $E_3^{*,0}(PG; \mathbb{F}_p)_{(\varpi)} = E_3^{*,0}(PG; \mathbb{F}_p) |_{\varpi=0}$. Lemma 4.2 implies that

Lemma 4.3. *In the order of $(G, p) = (SU(n), p)$ with $n = p^r n'$ and $(p, n') = 1$, $(Sp(n), 2)$ with $n = 2^r(2d+1)$, $(E_6, 3)$ and $(E_7, 2)$, one has*

- i) $J_p(\varpi) = \frac{\mathbb{F}_p[\varpi]}{\langle \varpi^{h(G)} \rangle}$ with $h(G) = p^r, 2^{r+1}, 9, 2$;
- ii) $E_3^{*,0}(PG; \mathbb{F}_p)_{(\varpi)} = \mathbb{F}_p, \mathbb{F}_2, \frac{\mathbb{F}_3[x_4]}{\langle x_4^3 \rangle}, \frac{\mathbb{F}_2[x_3, x_5, x_9]}{\langle x_3^2, x_5^2, x_9^2 \rangle};$
- iii) $E_3^{*,0}(PG; \mathbb{F}_p) = J_p(\varpi) \otimes E_3^{*,0}(PG; \mathbb{F}_p)_{(\varpi)}$,
- iv) the map C^* annihilates ϖ and carries the factor $1 \otimes E_3^{*,0}(PG; \mathbb{F}_p)_{(\varpi)}$
in iii) isomorphically onto the ring $E_3^{*,0}(G \times S^1; \mathbb{F}_p)$. \square

4.2 The ring $H^*(PG; \mathbb{F}_p)$

By Theorem 4.1 we can assume that $(G, p) = (SU(n), p)$ with $p \mid n$; $(Sp(n), 2)$; $(E_6, 3)$; $(E_7, 2)$. In view of the set $S_p(G)$ of primary characteristic polynomials of G over \mathbb{F}_p presented in Table 5.1 (see also ii) of Example 3.7) one obtains the degree set $D(G, p)$ of the polynomials in $S_p(G)$ as that tabulated below:

Table 4.1. The degree set of the primary characteristic polynomials over \mathbb{F}_p

(G, p)	$(SU(n), p)$	$(Spin(n), 2)$	$(E_6, 3)$	$(E_7, 2)$
$D(G, p)$	$\{2, 3, \dots, n\}$	$\{2, 4, \dots, 2n\}$	$\{2, 4, 5, 6, 8, 9\}$	$\{2, 3, 5, 8, 9, 12, 14\}$

For each $s \in D(G, p)$ let $\xi_{2s-1} := \varphi_p(P) \in E_3^{*,1}(G \times S^1, \mathbb{F}_p)$, where $P \in S_p(G)$ with $\deg P = s$. With the groups G being 1-connected one has by [8, Theorem 5.4] the ring isomorphism

$$E_3^{*,*}(G \times S^1, \mathbb{F}_p) = E_3^{*,0}(G \times S^1, \mathbb{F}_p) \otimes \Lambda(\xi_1, \xi_{2s-1})_{s \in D(G, p)},$$

where $\xi_1 \in E_3^{0,1}$ is the class specified in Theorem 3.9. Let $D(PG, p)$ be the complement of the number $h(G)$ (see i) of Lemma 4.3) in $D(G, p)$, put $B = E_3^{*,0}(G \times S^1, \mathbb{F}_p) \otimes \Lambda(\xi_1, \xi_{2s-1})_{s \in D(PG, p)}$ and rewritten

$$(4.3) \quad E_3^{*,*}(G \times S^1, \mathbb{F}_p) = B \oplus \xi_{2h(G)-1} \cdot B.$$

Then the exact sequence (3.13) takes the following useful form

$$(4.4) \quad 0 \rightarrow E_3^{*,*}(PG; \mathbb{F}_p)_{\langle \varpi \rangle} \xrightarrow{C^*} B \oplus \xi_{2h(G)-1} \cdot B \xrightarrow{\bar{\theta}} E_3^{*,*}(PG; \mathbb{F}_p) \xrightarrow{\pi} \langle \varpi \rangle \rightarrow 0.$$

With $p \mid q$ one has $\bar{\theta}(\xi_1) = 0$ by i) of Theorem 3.9. By (4.4) there is a unique class $\iota' \in E_3^{0,1}(G/\mathbb{Z}_q; \mathbb{F}_p) = \mathbb{F}_p$ such that

$$C^*(\iota') = \xi_1 \text{ (with } \kappa(\iota') = \iota, \widehat{\beta}_p(\iota') = \varpi \text{ by the diagram (3.5))}.$$

Furthermore, with the set $S_p(G)$ of characteristic polynomials for 1-forms ξ_{2s-1} being presented in Table 5.1, the formula (3.17) is applicable to evaluate $\bar{\theta}(\xi_{2s-1})$, $s \in D(G, p)$. The results recorded in Table 5.2 yield that

$$(4.5) \quad \bar{\theta}(\xi_{2s-1}) = 0 \text{ if } s \in D(PG, p), \omega^{h(G)-1} \text{ if } s = h(G).$$

With $\bar{\theta}(\xi_{2s-1}) = 0$ for $s \in D(PG, p)$ Lemma 3.11 assures us a polynomial $P' \in \langle \text{Im } \tilde{\tau} \rangle_p \cap \ker f_p$ with $\deg P' = s$ so that class $\zeta_{2s-1} = \varphi_p(P')$ satisfies

$$C^*(\zeta_{2s-1}) = \xi_{2s-1}.$$

A set of such polynomials P' so obtained, denoted by $S_p(PG)$, are presented in Table 5.3. Taking iv) of Lemma 4.3 into consideration the exactness of the sequence (4.4) forces the isomorphisms

$$\begin{aligned} E_3^{*,*}(PG; \mathbb{F}_p)_{\langle \omega \rangle} &\cong E_3^{*,0}(PG; \mathbb{F}_p)_{\langle \varpi \rangle} \otimes \Lambda(\iota', \zeta_{2s-1})_{s \in D(PG, p)}, \\ \xi_{2h(G)-1} \cdot B &\xrightarrow{\cong} \varpi^{h(G)-1} \cdot E_3^{*,*}(PG; \mathbb{F}_p)_{\langle \varpi \rangle} \text{ (by ii) of Theorem 3.9)} \end{aligned}$$

which imply that $\text{Im } \bar{\theta} = \langle \varpi^{h(G)-1} \rangle$. With the field \mathbb{F}_p as coefficient the exact sequence (3.12) is splittable. The formula (3.14) then yields the presentation

$$(4.6) \quad E_3^{*,*}(PG; \mathbb{F}_p) = E_3^{*,0}(PG; \mathbb{F}_p) \otimes \Lambda(\iota', \zeta_{2s-1})_{s \in D(PG, p)}.$$

By (4.6) the ring $E_3^{*,*}(PG; \mathbb{F}_p)$ is generated by $E_3^{*,0}$ and $E_3^{*,1}$, hence the differentials d_r act trivially on $E_r^{*,*}(PG; \mathbb{F}_p)$, $r \geq 4$. We get $E_3^{*,*}(PG; \mathbb{F}_p) = E_{\infty}^{*,*}(PG; \mathbb{F}_p)$. In particular, the maps π^* and κ in (3.3) and (3.4) are all monomorphisms. Combining this with the isomorphism of \mathbb{F}_p -spaces

$$E_\infty^{*,*}(PG; \mathbb{F}_p) = H^*(PG; \mathbb{F}_p)$$

we get from (4.6) the following additive presentation of the cohomology $H^*(PG; \mathbb{F}_p)$

$$(4.7) \quad H^*(PG; \mathbb{F}_p) = \pi^* E_3^{*,0}(PG; \mathbb{F}_p) \otimes \Delta(\iota, \zeta_{2s-1})_{s \in D(PG, p)},$$

where for simplicity the notion ζ_{2s-1} is reserved for the cohomology classes $\kappa(\zeta_{2s-1}) \in H^*(G; \mathbb{F}_p)$, and where one needs to replace exterior ring $\Lambda(\iota', \zeta_{2s-1})$ by the \mathbb{F}_p -module $\Delta(\iota, \zeta_{2s-1})$, since the properties $\iota'^2, \zeta_{2s-1}^2 = 0$ on $E_\infty^{*,*}(PG; \mathbb{F}_p)$ may not survive to $H^*(G; \mathbb{F}_p)$, see [7, Lemma 2.8].

Since $\zeta_{2s-1} = \kappa \circ \varphi_p(P)$ with $P \in S_p(PG)$ and $\deg P = s$, the formulae in Lemmas 3.7 and 3.8 is functional to compute $\beta_p(\zeta_{2s-1})$ and $Sq^{2r}(\zeta_{2s-1})$ in term of P . The calculation justifying the following results can be found in Section 5.

Lemma 4.4. *With respect to the presentation (4.7) one has*

- a) for $(G, p) = (SU(n), p)$ with $n = p^r n'$, $(n', p) = 1$
 $\beta_p(\zeta_{2s-1}) = -p^{r-t-1} \omega^{p^t}$ if $s = p^t$ with $t < r$, 0 otherwise
 (resp. $Sq^{2s-2} \zeta_{2s-1} = \zeta_{4s-3}$ for $2s-1 \leq 2^{r-1}$);
- b) for $(G, p) = (Sp(n), 2)$ with $n = 2^r(2b+1)$:
 $\beta_2(\zeta_{4s-1}) = \omega^{2^r}$ if $s = 2^{r-1}$, 0 if $s \neq 2^{r-1}$
 (resp. $Sq^{4s-2} \zeta_{4s-1} = \zeta_{8s-3}$ for $4s-1 \leq 2^r$);
- c) for $(G, p) = (E_6, 3)$ and in the order of $s = 2, 4, 5, 6, 8$
 $\beta_3(\zeta_{2s-1}) = 0, -x_4, 0, 0, -x_4^2$;
- d) for $(G, p) = (E_7, 2)$ and in the order of $s = 3, 5, 8, 9, 12, 14$
 $\beta_2(\zeta_{2s-1}) = x_3, x_5, x_3 x_5, x_9, x_3 x_9, x_5 x_9$
 (resp. $Sq^{2s-2} \zeta_{2s-1} = \zeta_9, \zeta_{17}, 0, 0, 0, 0$). \square

Combining (4.7) with Lemma 4.4 we show that

Theorem 4.5. *The rings $H^*(PG; \mathbb{F}_p)$ has the following presentations*

- i) $H^*(PSU(n); \mathbb{F}_2) = \frac{\mathbb{F}_2[\omega]}{\langle \omega^{2^r} \rangle} \otimes \Delta(\iota) \otimes \Lambda_{\mathbb{F}_2}(\zeta_3, \zeta_5, \dots, \widehat{\zeta}_{2^{r+1}-1}, \dots, \zeta_{2n-1})$,
 where $n = 2^r(2b+1)$, $\iota^2 = \omega$ or 0 in accordance to $r = 1$ or $r > 1$;
- ii) $H^*(PSU(n); \mathbb{F}_p) = \frac{\mathbb{F}_p[\omega]}{\langle \omega^{p^r} \rangle} \otimes \Lambda_{\mathbb{F}_p}(\iota, \zeta_3, \dots, \widehat{\zeta}_{2p^r-1}, \dots, \zeta_{2n-1})$,
 where $p \neq 2$, $n = p^r n'$ with $(n', p) = 1$;
- iii) $H^*(PSp(n); \mathbb{F}_2) = \frac{\mathbb{F}_2[\omega]}{\langle \omega^{2^{r+1}} \rangle} \otimes \Delta(\iota) \otimes \Lambda_{\mathbb{F}_2}(\zeta_3, \zeta_7, \dots, \widehat{\zeta}_{2^{r+2}-1}, \dots, \zeta_{4n-1})$,
 where $\iota^2 = \omega$, $n = 2^r(2b+1)$;
- iv) $H^*(PE_6; \mathbb{F}_3) = \frac{\mathbb{F}_3[\omega, x_4]}{\langle \omega^9, x_4^3 \rangle} \otimes \Lambda_{\mathbb{F}_3}(\iota, \zeta_3, \zeta_7, \zeta_9, \zeta_{11}, \zeta_{15})$;
- v) $H^*(PE_7; \mathbb{F}_2) = \frac{\mathbb{F}_2[\omega, x_3, x_5, x_9]}{\langle \omega^2, x_3^2, x_5^2, x_9^2 \rangle} \otimes \Delta(\iota, \zeta_5, \zeta_9) \otimes \Lambda_{\mathbb{F}_2}(\zeta_{15}, \zeta_{17}, \zeta_{23}, \zeta_{27})$,
 where $\iota^2 = \omega$, $\zeta_5^2 = x_5$, $\zeta_9^2 = x_9$.

Proof. In the presentations through i) to v) the first factor is $\text{Im } \pi^* \cong E_3^{*,0}(PG; \mathbb{F}_p)$, see iii) of Lemma 4.3. It remains to justify the expressions of the squares ι^2, ζ_{2s-1}^2 as that indicated in the theorem.

The cases ii) and iv) are trivial, as in a characteristic $p \neq 2$ the square of any odd degree cohomology class is zero. In the remaining cases i), iii) and v) we have $p = 2$. The relations $\iota^2 = \omega$ come from $\omega = \beta_q(\iota) \in H^2(PG)$ with $q = n, 2, 2$ in accordance to $G = SU(n), Sp(n), E_7$. To evaluate ζ_{2s-1}^2 with $s \in D(PG, p)$ we make use of the Steenrod operators Sq^{2r} by which

$$\zeta_{2s-1}^2 = \delta_2 \circ Sq^{2s-2}(\zeta_{2s-1}) \text{ (see [22])}.$$

Results in Lemma 4.4 implies that $\zeta_{2s-1}^2 = 0$ with the only exceptions $\zeta_5^2 = x_5, \zeta_9^2 = x_9$ when $(G, p) = (E_7, 2)$. This completes the proof. \square

4.3 The integral cohomology of $PSU(n), PSp(n)$

The integral cohomology of a space X admits a canonical decomposition

$$(4.8) \quad H^*(X) = \mathcal{F}(X) \bigoplus_p \sigma_p(X) \text{ with } \sigma_p(X) := \{x \in H^*(X) \mid p^r x = 0, r \geq 1\},$$

where the summands $\mathcal{F}(X)$ and $\sigma_p(X)$ are a *free part* and the p -*primary component* of $H^*(X)$, and where the sum is over all primes p . Therefore, the determination of the cohomology $H^*(X)$ essentially consists of two tasks:

- a) express $\mathcal{F}(X)$ and $\sigma_p(X)$ by explicit generators of the ring $H^*(X)$;
- b) decide the actions $\mathcal{F}(X) \times \sigma_p(X) \rightarrow \sigma_p(X)$ of the free part on $\sigma_p(X)$.

Keeping these in mind we compute the integral cohomology $H^*(PG)$ for $G = SU(n)$ and $Sp(n)$ by applying the exact sequence (1.8) (resp. (3.13)).

With the groups $G = SU(n)$ and $Sp(n)$ being 1-connected the integral cohomologies of the groups $G \times S^1$ are well known. Indeed, with respect to the degree set $D(G)$ of the set $S(G)$ of primary characteristic polynomials for G over \mathbb{Z} given in Table 4.4 (see also Example 3.6) one has

$$(4.9) \quad E_3^{*,*}(G \times S^1) = \Lambda(\gamma_1, \gamma_{2s-1})_{s \in D(G)} (\cong H^*(G \times S^1) \text{ via } \kappa \text{ in (3.4)}).$$

where $D(SU(n)) = \{2, \dots, n\}$, $D(Sp(n)) = \{4, \dots, 2n\}$, and where $\gamma_1 := \xi_1$, $\gamma_{2s-1} = \varphi(P) \in E_3^{*,1}(G \times S^1)$ with $P \in S(G)$ and $\deg P = s$. Moreover, in addition to the relations by i) of Theorem 3.9

$$\bar{\theta}(\gamma_1) = n \text{ or } 2 \text{ for } G = SU(n) \text{ or } Sp(n),$$

with $\gamma_{2s-1} = \varphi(P)$ the formula (3.17) is applicable to evaluate $\bar{\theta}(\gamma_{2s-1})$ in term of P . Explicitly, the computation recorded in Table 5.5 tells that

$$(4.10) \quad \bar{\theta}(\gamma_{2s-1}) = \begin{cases} \binom{n}{s} \varpi^{s-1} & \text{for } SU(n); \\ 0 & \text{if } s \neq 2^{r+1}, \varpi^{2^{r+1}-1} \text{ if } s = 2^{r+1}, \text{ for } Sp(2^r(2b+1)). \end{cases}$$

By the prime factorization of an integer $n \geq 2$ we mean the unique expression $n = p_1^{r_1} \cdots p_t^{r_t}$ with $1 < p_1 < \cdots < p_t$ the set of all prime factors of n . In term of this factorization one defines the partition on the set $\{2, \dots, n\}$ by

$$\{2, \dots, n\} = Q_0(n) \coprod_{1 \leq i \leq t} Q_{p_i}(n) \text{ with } Q_{p_i}(n) = \{p_i^r \mid 1 \leq r \leq r_s\}.$$

Lemma 4.6. Assume that $G = SU(n)$ with $n = p_1^{r_1} \cdots p_t^{r_t}$ (resp. $G = Sp(n)$ with $n = 2^r(2b+1)$). For a $s \in D(G)$ let $a_s \in \mathbb{Z}$ be the order of the class $\overline{\theta}(\gamma_{2s-1}) \in E_3^{*,0}(PG)$. Then

- i) $a_s = p_i$ or 1 for $s \in Q_{p_i}(n)$ or $s \in Q_0(n)$
(resp. $a_s = 2$ or 1 for $s = 2^{r+1}$ or $s \neq 2^{r+1}$);
- ii) there exists a class $\rho_{2s-1} \in E_3^{*,1}(PG)$ satisfying $C^*(\rho_{2s-1}) = a_s \gamma_{2s-1}$;
- iii) $\mathcal{F}(PG) = \Lambda(\rho'_{2s-1})_{s \in D(G)}$, where $\rho'_{2s-1} = \kappa(\rho_{2s-1}) \in H^*(PG)$.

Proof. For $SU(n)$ (resp. for $Sp(n)$) the relation i) comes from the formula (4.10) of $\overline{\theta}(\gamma_{2s-1})$, the presentation (4.2) of the ring $E_3^{*,0}(PSU(n))$ (resp. $E_3^{*,0}(PSp(n))$), as well as [4, Theorem 1.1] (see also (5.5)).

Property ii) follows from $\overline{\theta}(a_s \gamma_{2s-1}) = 0$, together with the exactness of the sequence (3.11).

To show iii) we set for a multi-index $I \subseteq D(G)$ that

$$a_I = \prod_{s \in I} a_s \in \mathbb{Z}, \gamma_I = \prod_{s \in I} \gamma_{2s-1} \in H^*(G), \rho_I = \prod_{s \in I} \rho'_{2s-1} \in H^*(PG).$$

For $G = SU(n)$ or $Sp(n)$ the degree of the covering $c : G \rightarrow PG$ is $\deg c = n$ or 2. On the other hand by (4.9) the monomial $\gamma_{D(G)}$ is a generator of the top degree cohomology group $H^m(G) = \mathbb{Z}$, $m = \dim G$, while the relations $a_{D(G)} = \deg c$, $C^*(\rho_{D(G)}) = a_{D(G)} \cdot \gamma_{D(G)}$ by i) and ii) implies that the monomial $\rho_{D(G)}$ is a generator of the group $H^m(PG) = \mathbb{Z}$. By [7, Lemma 2.9] the set $\{1, \rho_I \mid I \subseteq D(G)\}$ of monomials spans a direct summand of $\mathcal{F}(PG)$ with rank $2^{|D(G)|}$. It follows then from

$$\dim(\mathcal{F}(PG) \otimes \mathbb{Q}) = \dim(\mathcal{F}(G) \otimes \mathbb{Q}) = 2^{|D(G)|}$$

that the set $\{1, \rho_I \mid I \subseteq D(G)\}$ of monomials is a basis of $\mathcal{F}(PG)$.

The proof of iii) will be completed once we show that $\rho_{2s-1}^2 = 0$, $s \in D(G)$. Assume that $n = 2^r(2b+1)$. Since the square of an odd degree cohomology class belongs to $\text{Im } \beta_2$, and since the relation $\rho'_{2s-1} \in \text{Im } \kappa$ implies that $\rho_{2s-1}^2 \in \text{Im } \pi^*$ by [7, Lemma 2.8], we have

$$\rho_{2s-1}^2 \in \text{Im}[\pi^* \circ \widehat{\beta}_2 : E_3^{*,1}(PG; \mathbb{F}_2) \rightarrow E_3^{*,0}(PG) \rightarrow H^*(PG)].$$

On the other hand Lemma 4.3 implies that, if $G = SU(n)$ (resp. $G = Sp(n)$),

$$\text{Im } \pi^* \circ \widehat{\beta}_2 = \{0, 2^{r-t-1} \omega^{2^t} \mid 0 \leq t \leq r-1\} \text{ (resp. } \{0, \omega^{2^r}\}).$$

One obtains the desired relation $\rho_{2s-1}^2 = 0$ for the degree reason. \square

To simplify notation we reserve ρ_{2s-1} for ρ'_{2s-1} . Substituting the formula (4.9) for $H^*(G \times S^1)$ into (1.8) yields the following exact sequence, in which $H^*(PG)_{\langle \omega \rangle} = \Lambda(\rho_{2s-1})_{s \in D(G)}$ by the relations ii) and iii) of Lemma 4.6,

$$0 \rightarrow \Lambda(\rho_{2s-1})_{s \in D(G)} \xrightarrow{C^*} \Lambda(\gamma_1, \gamma_{2s-1})_{s \in D(G)} \xrightarrow{\theta} H^*(PG) \xrightarrow{\omega} \langle \omega \rangle \rightarrow 0.$$

Since the quotient group $H^*(PG)_{\langle \omega \rangle}$ is free the map g in the exact sequence (1.7) has a splitting homomorphism. Therefore, the formula (3.14) becomes functional to yield the presentation

$$(4.11) \quad H^*(PG) = \frac{J(\omega) \otimes \Lambda(\rho_{2s-1})_{s \in D(G)}}{\langle \omega \cdot \text{Im } \theta \rangle}, \text{ where}$$

$$(4.12) \quad J(\omega) = \begin{cases} \frac{\mathbb{Z}[\omega]}{\langle b_{n,r} \omega^r, 1 \leq r \leq n \rangle} & \text{for } G = SU(n); \\ \mathbb{Z} \oplus \frac{\mathbb{Z}[\omega]^+}{\langle 2\omega, \omega^{2r+1} \rangle} & \text{for } G = Sp(n) \text{ with } 2^r(2b+1) \end{cases} \quad \text{by (4.2).}$$

Theorem 4.7. *The integral cohomologies of the groups $PSp(n)$ ($n = 2^r(2b+1)$) and $PSU(n)$ ($n = p_1^{r_1} \cdots p_t^{r_t}$) are*

$$\text{i) } H^*(PSp(n)) = \Lambda(\rho_{4s-1})_{s \in \{1, \dots, n\}} \oplus \sigma_2(PSp(n)), \text{ where}$$

$$\sigma_2(PSp(n)) = \frac{\mathbb{Z}[\omega]^+ \otimes \Lambda(\rho_3, \rho_7, \dots, \rho_{4n-1})}{\langle \omega^{2r+1}, \omega \cdot \rho_{2r+2-1} \rangle}.$$

$$\text{ii) } H^*(PSU(n)) = \Lambda(\rho_{2s-1})_{s \in \{2, \dots, n\}} \bigoplus_{1 \leq s \leq t} \sigma_{p_s}(PSU(n)), \text{ where}$$

$$\sigma_{p_s}(PSU(n)) = \frac{\mathbb{Z}[\omega]^+ \otimes \Lambda(\rho_{2s-1})_{s \in \{2, \dots, n\}}}{\langle \omega \theta(\gamma_I) | I \subseteq \{1\} \amalg Q_{p_s}(n) \rangle}.$$

Proof. For both $G = Sp(n)$ and $SU(n)$ the presentation of the free part $\mathcal{F}(PG)$ stated in the theorem has been shown by iii) of Lemma 4.6. It remains to establish the formulae for the ideals $\sigma_p(PG)$.

If $G = Sp(n)$ the formulae (4.11) and (4.12) imply, in addition to $\sigma_p(PSp(n)) = 0$ when $p \neq 2$, that the map h in Theorem 1.3 restricts a surjection

$$h_2 : \mathbb{Z}[\omega]^+ \otimes \Lambda(\rho_3, \rho_7, \dots, \rho_{4n-1}) \rightarrow \sigma_2(PSp(n))$$

with $\ker h_2 = \langle \omega \cdot \text{Im } \theta(\gamma_I), \omega \cdot \text{Im } \theta(\gamma_1 \gamma_I) \mid I \subseteq D(Sp(n)) \rangle$. By formula (4.10) and the property ii) of Theorem 1.3

$$\begin{aligned} \theta(\gamma_I) &= 0 \text{ or } \omega^{2r} \rho_{I \setminus \{2^{r+1}\}} \text{ if } 2^{r+1} \notin I \text{ or } 2^{r+1} \in I; \\ \theta(\gamma_1 \gamma_I) &= 2\rho_I \text{ or } \rho_I \text{ if } 2^{r+1} \notin I \text{ or } 2^{r+1} \in I. \end{aligned}$$

Summarizing $\ker h_2 = \langle 2\omega, \omega^{2^{r+1}}, \omega \cdot \rho_{2^{r+2}-1} \rangle$, completing the proof of i).

For $G = SU(n)$ one can show that the ideal $\ker h = \langle \omega \cdot \text{Im } \theta \rangle$ admits the simplification

$$\text{a) } \ker h = \langle \omega \theta(\gamma_I) \mid I \subseteq \{1\} \amalg Q_{p_s}(n), 1 \leq s \leq t \rangle.$$

To show this we note that each multi-index $K \subseteq \{2, \dots, n\}$ has the partition

$$K = K_0 \bigsqcup_{1 \leq s \leq t} K_s \text{ with } K_0 = K \cap Q_0(n), K_s = K \cap Q_{p_s}(n).$$

Let $b_s = p_1^{|K_1|} \cdots \widehat{p_s^{|K_s|}} \cdots p_t^{|K_t|}$ with $|K_s|$ the cardinality of K_s . Since the set $\{b_1, \dots, b_t\}$ of integers is co-prime there is a set $\{q_1, \dots, q_t\}$ of integers satisfying $\sum q_s b_s = 1$. One obtains a) from the following calculation based on ii) of Theorem 1.3 and ii) of Lemma 4.6, where H_s is the complement of K_s in K :

$$\theta(\gamma_K) = \sum q_s \theta(b_s \gamma_K) = \sum q_s \rho_{H_s} \cup \theta(\gamma_{K_s});$$

$$\theta(\gamma_1 \gamma_K) = \sum q_s \theta(b_s \gamma_1 \gamma_K) = \sum q_s \rho_{H_s} \cup \theta(\gamma_1 \gamma_{K_s}).$$

On the other hand, by (4.12) the ring $J(\omega)$ admits the decomposition

$$b) \quad J(\omega) = \mathbb{Z} \bigoplus_{1 \leq s \leq t} J_s(\omega) \text{ with } J_s(\omega) = \frac{\mathbb{Z}[\omega]^+}{\langle p_s^{r_s} \omega, p_s^{r_s-1} \omega^{p_s}, p_s^{r_s-2} \omega^{p_s^2}, \dots, \omega^{p_s^{r_s}} \rangle}.$$

Substituting a) and b) into (4.11) and notice by ii) of Theorem 1.3 that

$$\theta(\gamma_I) \in \sigma_{p_s}(PSU(n)) \text{ for all } I \subseteq Q_{p_s}(n),$$

one gets, in addition to $\sigma_p(PSU(n)) = 0$ for $p \notin \{p_1, \dots, p_t\}$, the isomorphisms

$$c) \quad \frac{J_s(\omega) \otimes \Lambda(\rho_{2s-1})_{s \in \{1, \dots, n\}}}{\langle \omega \theta(\gamma_I) \mid I \subseteq \{1\} \amalg Q_{p_s}(n) \rangle} \cong \sigma_{p_s}(PSU(n)), \quad 1 \leq s \leq t. \square$$

In view of the formula c) for the ideal $\sigma_{p_s}(PSU(n))$ a complete description of the ring $H^*(PSU(n))$ asks for a formula expressing the classes $\theta(\gamma_I)$ with $I \subseteq \{1\} \amalg Q_{p_s}(n)$ by the generators ω and ρ_{2s-1} of the ring $H^*(PSU(n))$. We emphasize at this point that presently the classes ρ_{2s-1} are only determined modulo the ideal $\langle \omega \rangle$, while formulae for $\theta(\gamma_I)$ may vary with respect to different choices of the classes ρ_{2s-1} . With an appropriate choice of the primary 1-forms ρ_{2s-1} by their characteristic polynomials (see formula (5.8)) the following result will be established in Section 5.4.

Proposition 4.8. *If $n = p^r$ the classes ρ_{2p^s-1} , $1 \leq s \leq r$, may be chosen so that, for an $I = \{p^{i_1}, \dots, p^{i_k}\} \in \{1\} \amalg Q_p(n)$ with $0 \leq i_1 < \dots < i_k \leq r$,*

i) *the class $\theta(\gamma_I) \in H^*(PSU(n))$ is divisible by p whenever $i_k < r$;*

ii) *$\theta(\gamma_{2p^s-1}) = p^{r-s}$ if $I = \{p^s\}$ is a singleton;*

iii) *$\theta(\gamma_I) = (\frac{1}{p} \theta(\gamma_{I^e})) \cdot \rho_{2p^{i_k}-1} + (\frac{1}{p} \theta(\gamma_{I^\partial})) \cdot \omega^{p^{i_k}-p^{i_k-1}}$*

where $I^e = \{p^{i_1}, \dots, p^{i_{k-1}}\}$, $I^\partial = \{p^{i_1}, \dots, p^{i_{k-1}}\}$. \square

In view of the formula c) the assumption $n = p^r$ in Proposition 4.8 may not be necessary for the following reason. If $n = p^r n'$ with $(p, n') = 1$ then

$$Q_p(n) = Q_p(p^r).$$

It indicates that, with respect to the cohomology ring map induced by the inclusion $PSU(p^r) \rightarrow PSU(n)$, the set $\langle \omega \theta(\gamma_I) \mid I \subseteq \{1\} \amalg Q_p(n) \rangle$ of relations on $\sigma_p(PSU(n))$ are in one to one correspondence with that on $\sigma_p(PSU(p^r))$, where the latter are handled by Proposition 4.8.

Example 4.9. In Proposition 4.8 we notice that

a) if $i_k = i_{k-1} + 1$ then $\gamma_{I^\partial} = 0$ by the relation $\gamma_{2r-1}^2 = 0$ on $H^*(U(n))$;

b) the classes $\theta(\gamma_{I^e})$ and $\theta(\gamma_{I^\partial})$ are always divisible by p by ii).

Therefore, formula iii) gives an effective recurrence to evaluate $\theta(\gamma_I)$. As examples when $n = 2^3$ we get that

$$\begin{aligned} \theta(\gamma_{\{1,2,4\}}) &= 2\rho_3\rho_7; \\ \theta(\gamma_{\{1,2,8\}}) &= 2\rho_3\rho_{15} + \omega^4\rho_3\rho_7; \\ \theta(\gamma_{\{1,4,8\}}) &= 2\rho_7\rho_{15} + \omega^2\rho_3\rho_{15}; \\ \theta(\gamma_{\{2,4,8\}}) &= \omega\rho_7\rho_{15}; \\ \theta(\gamma_{\{1,2,4,8\}}) &= \rho_3\rho_7\rho_{15}. \square \end{aligned}$$

4.4 The integral cohomology of the groups PE_6 and PE_7

For a space X and a prime p the pair $\{H^*(X; \mathbb{F}_p); \delta_p\}$ with $\delta_p = r_p \circ \beta_p$ is a cochain complex whose cohomology $\overline{H}^*(X; \mathbb{F}_p)$ is the mod p Bockstein cohomology of X . For $(G, p) = (E_6, 3)$ and $(E_7, 2)$ the complexes $\{H^*(PG; \mathbb{F}_p); \delta_p\}$ have been decided by Lemma 4.4 and Theorem 4.5. Explicitly we have

$$(4.13a) \quad H^*(PE_6; \mathbb{F}_3) = [\text{Im } \pi^* \otimes \Lambda_{\mathbb{F}_3}(\varsigma_1, \varsigma_7)] \otimes \Lambda_{\mathbb{F}_3}(\varsigma_{2s-1})_{s \in \{2, 5, 6, 8\}} \text{ with}$$

$$\text{i) } \text{Im } \pi^* = \frac{\mathbb{F}_3[\omega, x_4]}{\langle \omega^9, x_4^3 \rangle}; \text{ ii) } \delta_3(\varsigma_1) = \omega, \delta_3(\varsigma_7) = x_4, \delta_2(\varsigma_{2s-1}) = 0,$$

where in view of iv) of Theorem 4.5 and in the order of $r = 1, 2, 4, 5, 6, 8$

$$\varsigma_{2r-1} := \iota, \varsigma_3, \varsigma_7, \varsigma_9, \varsigma_{11}, \varsigma_{15} - x_4 \varsigma_7.$$

$$(4.13b) \quad H^*(PE_7; \mathbb{F}_2) = [\text{Im } \pi^* \otimes \Delta_{\mathbb{F}_2}(\varsigma_1, \varsigma_{2t-1})_{t \in \{3, 5, 9\}}] \otimes \Lambda_{\mathbb{F}_2}(\varsigma_{2s-1})_{s \in \{8, 12, 14\}} \text{ with}$$

$$\text{i) } \text{Im } \pi^* = \frac{\mathbb{F}_2[\omega, x_3, x_5, x_9]}{\langle x_1^2, x_3^2, x_5^2, x_9^2 \rangle}; \text{ ii) } \delta_3(\varsigma_1) = \omega, \delta_3(\varsigma_{2t-1}) = x_t, \delta_3(\varsigma_{2s-1}) = 0,$$

where in view of v) of Theorem 4.5 and in the order of $r = 1, 3, 5, 8, 9, 12, 14$

$$\varsigma_{2r-1} := \iota, \varsigma_5, \varsigma_9, \varsigma_{15} + x_3 \varsigma_9, \varsigma_{17}, \varsigma_{23} + x_3 \varsigma_{17}, \varsigma_{27} + x_5 \varsigma_{17}.$$

In what follows we put $c_{\{1, 4\}} = \delta_3(\varsigma_1 \varsigma_7) \in H^9(PE_6; \mathbb{F}_3)$. For a multi-index $I \subseteq \{1, 3, 5, 9\}$ define the elements

$$c_I = \delta_2(\varsigma_I) \in H^*(PE_7; \mathbb{F}_2) \text{ with } \varsigma_I = \prod_{s \in I} \varsigma_{2s-1},$$

Lemma 4.10. *The cohomology $\overline{H}^*(PG; \mathbb{F}_p)$, together with $\text{Im } \delta_p$, are given by*

$$\text{i) } \overline{H}^*(PE_6; \mathbb{F}_3) \cong \Lambda_{\mathbb{F}_3}(\omega^8 \varsigma_1, x_4^2 \varsigma_7, \varsigma_3, \varsigma_9, \varsigma_{11}, \varsigma_{15})$$

$$\text{Im } \delta_3 = \frac{\mathbb{F}_3[\omega, x_4, c_{\{1, 4\}}]^+}{\langle \omega^9, x_4^3, c_{\{1, 4\}}^2, x_1^8 x_4^2 c_{\{1, 4\}} \rangle} \otimes \Lambda_{\mathbb{F}_3}(\varsigma_3, \varsigma_9, \varsigma_{11}, \varsigma_{15})$$

$$\text{ii) } \overline{H}^*(PE_7; \mathbb{F}_2) \cong \Lambda_{\mathbb{F}_2}(\omega \varsigma_1, x_3 \varsigma_5, x_5 \varsigma_9, x_9 \varsigma_{17}, \varsigma_{15}, \varsigma_{23}, \varsigma_{27})$$

$$\text{Im } \delta_2 = \frac{\mathbb{F}_2[\omega, x_3, x_5, x_9, c_I]^+}{\langle \omega^2, x_3^2, x_5^2, x_9^2, D_I, R_I, S_{I, J} \rangle} \otimes \Lambda_{\mathbb{F}_2}(\varsigma_{15}, \varsigma_{23}, \varsigma_{27}) \text{ with } |I|, |J| \geq 2,$$

where in the presentation of $\text{Im } \delta_2$ the relations $D_I, R_I, S_{I, J}$ are, respectively,

$$(4.14) \quad \sum_{t \in I} x_t c_{I_t} = 0, \quad \left(\prod_{t \in I} x_t \right) c_I = 0, \quad c_I c_J + \sum_{t \in I} x_t \prod_{s \in I_t \cap J} \varsigma_{2s-1}^2 c_{\langle I_t, J \rangle} = 0,$$

and where $(\varsigma_1^2, \varsigma_5^2, \varsigma_9^2, \varsigma_{17}^2) = (\omega, x_5, x_9, 0)$, I_t is the complement of $t \in I$, $\langle I, J \rangle$ denotes the complement of the intersection $I \cap J$ in the union $I \cup J$.

Proof. The results in i) and ii) come from the same calculation. We may therefore focus on the relatively nontrivial case ii). In the presentation (4.13b) of $H^*(PE_7; \mathbb{F}_2)$ the first factor $\text{Im } \pi^* \otimes \Delta_{\mathbb{F}_2}(\varsigma_1, \varsigma_{2t-1})$ is the Koszul complex studied in [7, Theorem 2.1], while the differential δ_2 acts trivially on the second factor $\Lambda_{\mathbb{F}_2}(\varsigma_{2s-1})$. One gets by [7, Theorem 2.1] and the Künneth formula that

$$\text{c) } \overline{H}^*(PE_7; \mathbb{F}_2) = \Delta_{\mathbb{F}_2}(x_1 \varsigma_1, x_3 \varsigma_5, x_5 \varsigma_9, x_9 \varsigma_{17}) \otimes \Lambda_{\mathbb{F}_2}(\varsigma_{15}, \varsigma_{23}, \varsigma_{27}),$$

$$\text{d) } \text{Im } \delta_2 = \frac{\text{Im } \pi^* \{1, c_I\}^+}{\langle D_J, R_K \rangle} \otimes \Lambda_{\mathbb{F}_2}(\varsigma_{15}, \varsigma_{23}, \varsigma_{27}),$$

where $I, J, K \subseteq \{1, 3, 5, 9\}$ with $|I|, |J|, |K| \geq 2$. By v) of Theorem 4.5 the formula c) of $\overline{H}^*(PE_7; \mathbb{F}_2)$ is identical to the one stated in ii). To modify the additive presentation of $\text{Im } \delta_2$ in d) into its ring presentation in ii) one needs to clarify the multiplicative rule among the classes c_I 's. This brings us the relations of the type $S_{I,J}$ which are obtained by the following calculation:

$$\begin{aligned} c_I c_J &= \delta_2(\zeta_I) \delta_2(\zeta_J) = \delta_2(\delta_2(\zeta_I) \zeta_J) \text{ (since } \delta_2^2 = 0) \\ &= \delta_2\left(\sum_{t \in I} x_t \zeta_{I_t} \zeta_J\right) \text{ (since } \delta_2(\zeta_I) = \sum_{t \in I} x_t \zeta_{I_t}) \\ &= \delta_2\left(\sum_{t \in I} x_t \prod_{s \in I_t \cap J} \zeta_{2s-1}^2 \zeta_{\langle I_t, J \rangle}\right) \text{ (with } \prod_{s \in I \cap J} \zeta_{2s-1}^2 = 1 \text{ if } I \cap J = \emptyset) \\ &= \sum_{t \in I} x_t \prod_{s \in I_t \cap J} \zeta_{2s-1}^2 c_{\langle I_t, J \rangle} \text{ (since } \delta_2(x_t) = 0, \delta_2(\zeta_{2s-1}^2) = 0\text{). } \square \end{aligned}$$

The presentations of the Bockstein cohomology $\overline{H}^*(PG; \mathbb{F}_p)$ in Lemma 4.10 provide us with crucial information on reduction $r_p : H^*(PG) \rightarrow H^*(PG; \mathbb{F}_p)$ for $(G, p) = (E_6, 3)$ and $(E_7, 2)$. Indeed, in view of the decomposition (4.8) for $X = PG$ we write r_p^0 and r_p^1 for the restrictions of r_p on $\mathcal{F}(PG)$ and $\sigma_p(PG)$, respectively. With

$$\dim \overline{H}^*(PE_6; \mathbb{F}_3) = 2^6 \text{ and } \dim \overline{H}^*(PE_7; \mathbb{F}_2) = 2^7$$

by Lemma 4.10 the result [7, Theorem 2.7] implies that

$$(4.15) \quad r_p^1 \text{ is an isomorphism } \sigma_p(PG) \cong \text{Im } \delta_p;$$

$$(4.16) \quad r_p^0 \text{ induces an isomorphism } \mathcal{F}(PG)/p \cdot \mathcal{F}(PG) \rightarrow \overline{H}^*(PG; \mathbb{F}_p).$$

To apply the exact sequence (1.8) to compute $H^*(PG)$ we need information on the cohomology of the group G . In term of the set $S(G)$ of primary characteristic polynomials for $G = E_6$ and E_7 over \mathbb{Z} given in Table 5.4 (see also Example 3.6) let $D(G)$ be the degree set of elements in $S(G)$. That is

$$D(E_6) = \{2, 5, 6, 8, 9, 12\}; D(E_7) = \{2, 6, 8, 10, 12, 14, 18\}$$

For each $s \in D(G)$ we set $\gamma_{2s-1} = \varphi(P) \in E_3^{*,1}(G)$, where $P \in S(G)$ with $\deg P = s$. To save notations we maintain γ_{2s-1} for $\kappa(\gamma_{2s-1}) \in H^*(G)$. By [7, Theorem 1.9] the integral cohomology $H^*(G)$ has the following presentations

$$(4.17a) \quad H^*(E_6) = \Delta(\gamma_3) \otimes \Lambda(\gamma_9, \gamma_{11}, \gamma_{15}, \gamma_{17}, \gamma_{23}) \oplus \sigma_2(E_6) \oplus \sigma_3(E_6) \text{ with}$$

$$\sigma_2(E_6) = \mathbb{F}_2[x_3]^+ / \langle x_3^2 \rangle \otimes \Delta(\gamma_3) \otimes \Lambda(\gamma_9, \gamma_{15}, \gamma_{17}, \gamma_{23}),$$

$$\text{where } \gamma_3^2 = x_3, x_3 \gamma_{11} = 0, x_4 \gamma_{23} = 0.$$

$$(4.17b) \quad H^*(E_7) = \Delta(\gamma_3) \otimes \Lambda_{\mathbb{Z}}(\gamma_{11}, \gamma_{15}, \gamma_{19}, \gamma_{23}, \gamma_{27}, \gamma_{35}) \bigoplus_{p=2,3} \sigma_p(E_7) \text{ with}$$

$$\sigma_3(E_7) = \frac{\mathbb{F}_3[x_4]^+}{\langle x_4^3 \rangle} \otimes \Lambda(\gamma_3, \gamma_{11}, \gamma_{15}, \gamma_{19}, \gamma_{27}, \gamma_{35}),$$

$$\text{where } \gamma_3^2 = x_3, x_4 \gamma_{23} = 0, r \in \{11, 19, 35\},$$

where the classes x_i 's are the special Schubert classes on E_n/T , $n = 6, 7$, specified in (5.1), and where the formulae of the ideals $\sigma_3(E_6)$ and $\sigma_2(E_7)$ are not needed in sequel, hence are omitted.

With $\gamma_{2s-1} = \kappa\varphi(P)$ one computes $\theta(\gamma_{2s-1})$ by the formula (3.17) to yield

(4.18) $\theta(\gamma_{2s-1}) = 0$ with the only exceptions: $\theta(\gamma_{2s-1}) = \omega^8$ or ω for $(G, s) = (E_6, 9)$ or $(E_7, 2)$ (see the contents of Table 5.5).

In view of the presentations (4.13a) and (4.13b) introduce the elements

$$\mathcal{C}_{\{1,4\}} = \beta_3(\varsigma_1 \varsigma_7) \in \sigma_3(PE_6), \mathcal{C}_K = \beta_2\left(\prod_{t \in K} \varsigma_{2t-1}\right) \in \sigma_2(E_7),$$

where $K \subseteq \{1, 3, 5, 9\}$. For $I, J \subset \{1, 3, 5, 9\}$ with $|I|, |J| \geq 2$ let $\mathcal{D}_I, \mathcal{R}_I, \mathcal{S}_{I,J} \in \sigma_2(E_7)$ be respectively the elements obtained by substituting in the polynomials $D_I, R_I, S_{I,J} \in \mathbb{F}_2[\omega, x_3, x_5, x_9, c_I]^+$ in (4.14) the classes c_I by C_I .

Lemma 4.11. *For each $s \in D(G)$ there exists a class $\rho_{2s-1} \in \text{Im } \kappa$ satisfying*

i) $C^*(\rho_{2s-1}) = a_s \cdot \gamma_{2s-1}$, where $a_s = 1$ with the only exceptions:

$a_9 = 3$ for E_6 ; $a_2 = 2$ for E_7 .

ii) $\mathcal{F}(PG) = \begin{cases} \Delta(\rho_3) \otimes \Lambda(\rho_9, \rho_{11}, \rho_{15}, \rho_{17}, \rho_{23}) \text{ with } \rho_3^2 = x_3 \text{ for } G = E_6; \\ \Lambda(\rho_3, \rho_{11}, \rho_{15}, \rho_{19}, \rho_{23}, \rho_{27}, \rho_{35}) \text{ for } G = E_7. \end{cases}$

iii) $\sigma_p(PG) = \begin{cases} \frac{\mathbb{F}_3[\omega, x_4, \mathcal{C}_{\{1,4\}}]^+ \otimes \Lambda(\rho_3, \rho_9, \rho_{11}, \rho_{15}, \rho_{17})}{\langle \omega^9, x_4^3, \omega \cdot \rho_{17}, \mathcal{C}_{\{1,4\}}^2, \omega^8 x_4^2 \mathcal{C}_{\{1,4\}} \rangle} \text{ for } (G, p) = (E_6, 3); \\ \frac{\mathbb{F}_2[\omega, x_3, x_5, x_9, \mathcal{C}_I]^+}{\langle x_1^2, x_3^2, x_5^2, x_9^2, \mathcal{D}_I, \mathcal{R}_I, \mathcal{S}_{I,J} \rangle} \otimes \Lambda(\rho_{15}, \rho_{23}, \rho_{27}) \text{ for } (G, p) = (E_7, 2). \end{cases}$

Proof. The classes $\rho_{2s-1} \in \text{Im } \kappa$ are specified by $\theta(a_s \gamma_{2s-1}) = 0$ by (4.18), as well as the exact sequence (1.8). This shows i).

For ii) the degree of the covering $c : G \rightarrow PG$ satisfies that $\deg c = a_{D(D)}$ ($= 3$ or 2 for E_6 or E_7) by i). The same argument as that used in the proof of iii) of Lemma 4.6 shows that the set $\{1, \rho_I \mid I \subseteq D(G)\}$ of monomials is a basis of $\mathcal{F}(PG)$. Since $\rho_{2s-1} \in \text{Im } \kappa$ and since the reduction r_p preserves $\text{Im } \kappa$ we get from (4.16) and the formula of $\overline{H}^*(PG; \mathbb{F}_p)$ in Lemma 4.10 the relations

$$(4.19a) \quad r_3(\rho_{2s-1}) \in \{\omega^8 \varsigma_1, x_4^2 \varsigma_7, \varsigma_3, \varsigma_9, \varsigma_{11}, \varsigma_{15}\} \text{ for } PE_6;$$

$$(4.19b) \quad r_2(\rho_{2s-1}) \in \{\omega \varsigma_1, x_3 \varsigma_5, x_5 \varsigma_9, x_9 \varsigma_{17}, \varsigma_{15}, \varsigma_{23}, \varsigma_{27}\} \text{ for } PE_7.$$

These determine the classes $r_p(\rho_{2s-1}) \in H^*(PG; \mathbb{F}_p)$ for the degree reason.

To complete the proof of ii) it remains to show that $\rho_{2s-1}^2 = 0$ with the only exception $\rho_3^2 = x_3$ when $G = E_6$. For PE_7 this comes from $\rho_{2s-1}^2 \in \sigma_2(PE_7)$, the injectivity of r_2 on $\sigma_2(PE_7)$ by (4.15), as well as the relation (4.19b). For PE_6 we shall show in the proof of Theorem 4.12 that the map C^* restricts to a ring monomorphism $\sigma_2(PE_6) \rightarrow \sigma_2(E_6 \times S^1)$. One obtains i) from

$$C^*(\rho_{2s-1}) \equiv \gamma_{2s-1} \pmod{2}, \quad C^*(x_3) \equiv x_3 \pmod{2} \text{ (see ii) of Lemma 4.2),}$$

as well as the relations $\gamma_3^2 = x_6, \gamma_{2s-1}^2 = 0, s \geq 3$, on the ring $H^*(E_6)$ by (4.19a).

Finally, granted with the isomorphisms r_p^1 in (4.15), the relations (4.19a) and (4.19b) suffice to translate the presentations of $\text{Im } \delta_p$ in Lemma 4.10 into the formulae for the ideals $\sigma_p(PG)$ stated in iii). \square

We present the integral cohomology $H^*(PG)$ with $G = E_6$ and E_7 by the set $\{\rho_{2s-1}\} \subset \text{Im } \kappa$ of 1-forms specified by i) of Lemma 4.11, the subring $\text{Im } \pi^*$ given by (4.2), together the torsion classes \mathcal{C}_K .

Theorem 4.12. *The rings $H^*(PG)$ with $G = E_6$ and E_7 are*

i) $H^*(PE_6) = \Delta(\rho_3) \otimes \Lambda(\rho_9, \rho_{11}, \rho_{15}, \rho_{17}, \rho_{23}) \bigoplus_{p=2,3} \sigma_p(PE_6)$ with

$$\sigma_2(PE_6) = \mathbb{F}_2[x_3]^+ / \langle x_3^2 \rangle \otimes \Delta(\rho_3) \otimes \Lambda(\rho_9, \rho_{15}, \rho_{17}, \rho_{23}),$$

$$\sigma_3(PE_6) = \frac{\mathbb{F}_3[\omega, x_4, \mathcal{C}_{\{1,4\}}]^+ \otimes \Lambda(\rho_3, \rho_9, \rho_{11}, \rho_{15}, \rho_{17})}{\langle \omega^9, x_4^3, \omega \cdot \rho_{17}, \mathcal{C}_{\{1,4\}}^2, \omega^8 x_4^2 \mathcal{C}_{\{1,4\}} \rangle},$$

that are subject the relations

$$\rho_3^2 = x_3, x_3 \rho_{11} = 0, x_4 \rho_{23} = 0, \omega \rho_{23} = x_4^2 \mathcal{C}_{\{1,4\}}, \mathcal{C}_{\{1,4\}} \rho_{23} = 0.$$

ii) $H^*(PE_7) = \Lambda(\rho_3, \rho_{11}, \rho_{15}, \rho_{19}, \rho_{23}, \rho_{27}, \rho_{35}) \bigoplus_{p=2,3} \sigma_p(PE_7)$ with

$$\sigma_2(PE_7) = \frac{\mathbb{F}_2[\omega, x_3, x_5, x_9, \mathcal{C}_I]^+}{\langle x_1^2, x_3^2, x_5^2, x_9^2, \mathcal{D}_I, \mathcal{R}_I, \mathcal{S}_I, J \rangle} \otimes \Lambda(\rho_{15}, \rho_{23}, \rho_{27});$$

$$\sigma_3(PE_7) = \frac{\mathbb{F}_3[x_4]^+}{\langle x_4^3 \rangle} \otimes \Lambda(\rho_3, \rho_{11}, \rho_{15}, \rho_{19}, \rho_{27}, \rho_{35}),$$

that are subject to the relations, where $K \subseteq \{1, 3, 5, 9\}$, $s \in \{2, 6, 10, 18\}$,

$$x_4 \rho_{23} = 0, \rho_{2s-1} C_K = 0 \text{ if } s \in K, \rho_{2s-1} C_K = x_{\frac{s}{2}} C_{K \cup \{s\}} \text{ if } s \notin K.$$

Proof. The presentations of $\mathcal{F}(PG)$, together with the ideals $\sigma_3(PE_6)$ and $\sigma_2(PE_7)$, have been shown by Lemma 4.11. Moreover, with $\langle \omega \rangle \in \sigma_3(PE_6)$ by $3\omega = 0$ ($\langle \omega \rangle \in \sigma_2(PE_7)$ by $2\omega = 0$) formula (4.8) yields that

$$H^*(PE_6)_{\langle \omega \rangle} = \mathcal{F}(PE_6) \bigoplus_{p \neq 3} \sigma_p(PE_6) \oplus \sigma_3(PE_6) / \langle \omega \rangle$$

$$(\text{resp. } H^*(PE_7)_{\langle \omega \rangle} = \mathcal{F}(PE_7) \bigoplus_{p \neq 2} \sigma_p(PE_7) \oplus \sigma_2(PE_6) / \langle \omega \rangle).$$

In particular, the map C^* in (1.8) restricts to the monomorphisms

$$\sigma_p(PE_6) \rightarrow \sigma_p(E_6 \times S^1), p \neq 3 \quad (\sigma_p(PE_7) \rightarrow \sigma_p(E_7 \times S^1), p \neq 2).$$

It implies, in addition to $\sigma_p(PG) = 0$ for all $p \neq 2, 3$, that

$$\sigma_2(PE_6) \cong \sigma_2(E_6) \quad (\text{resp. } \sigma_3(PE_7) \cong \sigma_3(E_7)) \text{ under } C^*.$$

One obtains from (4.17a) (resp. (4.17b)) the presentation of $\sigma_2(PE_6)$ (resp. $\sigma_3(PE_7)$), together with the relation $x_3 \rho_{11} = 0$ (resp. $x_4 \rho_{23} = 0$), as that stated in the theorem.

Finally, granted with the isomorphism r_p^1 in (4.15), Theorem 4.5, as well as the relations (4.19a) and (4.19b) that characterize the reduction $r_p^1 : \mathcal{F}(PG) \rightarrow H^*(PG; \mathbb{F}_p)$, one verifies the following relations

$$\text{a) } x_4 \rho_{23} = 0, \omega \rho_{23} = x_4^2 \mathcal{C}_{\{1,4\}}, \mathcal{C}_{\{1,4\}} \rho_{23} = 0;$$

$$\text{b) } x_4 \rho_{23} = 0, \rho_{2s-1} C_K = 0 \text{ if } s \in K, = x_{\frac{s}{2}} C_{I \cup \{s\}} \text{ if } s \notin K.$$

that characterize the action of $\mathcal{F}(PG)$ on $\sigma_p(PG)$, respectively for $(G, p) = (E_6, 3)$ or $(E_7, 2)$. As example, combining (4.15) and iv) of Theorem 4.5, the relations in a) are verified by the following calculations in the ring $H^*(PE_6; \mathbb{F}_3)$

$$r_3(\omega \rho_{23}) = \omega x_4^2 \zeta_7 = x_4^2 (\omega \zeta_7 - \iota x_4) = x_4^2 \mathcal{C}_{\{1,4\}};$$

$$r_3(x_4 \rho_{23}) = x_4^3 \zeta_7 = 0; r_3(\mathcal{C}_{\{1,4\}} \rho_{23}) = (\omega \zeta_7 - \iota x_4) x_4^2 \zeta_7 = 0,$$

where $r_3(\rho_{23}) = x_4^2 \zeta_7$ by (4.19a). \square

4.5 Historical remarks

The rings $H^*(PG; \mathbb{F}_p)$ concerned by Theorem 4.5 have been previously computed by Baum and Browder [1] for $G = SU(n), Sp(n)$, and by Toda, Kono and Ishitoya [11, 13] for $G = E_6, E_7$. However, with respect to our explicitly constructed generators on $H^*(PG; \mathbb{F}_p)$ the Bockstein operator β_p can be effectively calculated, which played a crucial role to decide the torsion ideals $\sigma_p(PG)$ of the integral cohomology $H^*(PG)$.

In [20, Theorem A] Ruiz stated a presentation of the integral cohomology ring of the projective complex Stiefel manifold $Y_{n,n-m}$. It implies when taking $m = 0$ that the map h in Theorem 1.3 is an isomorphism for the group $PSU(n)$. That is, the relations of the form $\omega\theta(\gamma_I)$ with $|I| \geq 2$ are absent. Computation in Example 4.9 indicates that these missing relations are highly nontrivial.

The formula (2.9) for the Borel transgression τ is an essential ingredient for the computation throughout Section 4. On the other hand in [12, formula (4)] Kač stated a formula for the differential d_2 on $E_2^{*,*}(G; \mathbb{F}_p)$ which implies that the map τ is an isomorphism for any semi-simple Lie group G and characteristic p . These suggest to us that a proof of Theorem 2.4 is unavoidable.

According Grothendieck [9] the subring $\text{Im } \pi^* \subset H^*(PG)$ is the *Chow ring* $A^*(PG^c)$ of the reductive algebraic group PG^c corresponding to PG . In this regard the formulae (4.2) presents the Chow rings $A^*(PG^c)$ for $G = SU(n), Sp(n), E_6, E_7$ by explicit Schubert classes on G/T .

It has been known for a long time that for cohomology with coefficients in a field \mathbb{F} one has $E_3^{*,*}(G; \mathbb{F}) = H^*(G; \mathbb{F})$ [12, 19]. For the integral cohomology the maps π^* and κ in (3.5) and (3.6) provide a direct passage from $E_3^{*,*}(G)$ to $H^*(G)$ useful to show the much stronger relation

$$(4.20) \quad E_3^{*,*}(G) = H^*(G).$$

Indeed, for the 1-connected Lie groups G the relation (4.20) was conjecture by Marlin [16] and has been confirmed by the authors in [7, Theorem 3.6], while our proofs of Theorem 4.7 and Theorem 4.12 indicate that the relation (4.20) holds more generally by all compact Lie groups.

5 Schubert calculus

This section generates and records the intermediate data facilitating the computation in Section 4. It serves also the purpose to illustrate how the construction and computation with the cohomology of compact Lie groups G can be boiling down to computing with certain polynomials in the Schubert classes on G/T .

5.1 Schubert presentation of the ring $H^*(G/T)$

Let $m = n - 1, n, 6$ or 7 in accordance to $G = SU(n), Sp(n), E_6$ or E_7 . In view of the Schubert basis $\{\omega_1, \dots, \omega_m\}$ on $H^2(G/T)$ (see Lemma 2.3) we define the polynomials $c_r(G) \in H^*(BT) = \mathbb{Z}[\omega_1, \dots, \omega_m]$ to be the r^{th} elementary symmetric polynomials on the set $\Omega(G)$ specified below

$$\begin{aligned} \Omega(SU(n)) &= \{\omega_1, \omega_k - \omega_{k-1}, -\omega_{n-1} \mid 2 \leq k \leq n-1\}; \\ \Omega(Sp(n)) &= \{\pm\omega_1, \pm(\omega_k - \omega_{k-1}) \mid 2 \leq k \leq n\}; \end{aligned}$$

$$\begin{aligned}\Omega(E_6) &= \{\omega_6, \omega_5 - \omega_6, \omega_4 - \omega_5, \omega_2 + \omega_3 - \omega_4, \omega_1 + \omega_2 - \omega_3, \omega_2 - \omega_1\}; \\ \Omega(E_7) &= \{\omega_7, \omega_6 - \omega_7, \omega_5 - \omega_6, \omega_4 - \omega_5, \omega_2 + \omega_3 - \omega_4, \omega_1 + \omega_2 - \omega_3, \omega_2 - \omega_1\}.\end{aligned}$$

Using the *Weyl coordinates* $\sigma_{[i_1, \dots, i_k]}$ for the Schubert classes on G/T ([7, Definition 2]) we introduce also the *special Schubert classes* x_r on G/T by

$$\begin{aligned}(5.1) \quad (x_3, x_4) &:= (\sigma_{[5,4,2]}, \sigma_{[6,5,4,2]}) \text{ for } G = E_6; \\ (x_3, x_4, x_5, x_9) &:= (\sigma_{[5,4,2]}, \sigma_{[6,5,4,2]}, \sigma_{[7,6,5,4,2]}, \sigma_{[1,5,4,3,7,6,5,4,2]}) \text{ for } G = E_7.\end{aligned}$$

With these notation we have by [6] the following presentations of the cohomologies $H^*(G/T)$ in term of explicit generators and relations, where in the cases of $G = E_6$ and E_7 and in comparison with Theorem 2.6, we use $x_{\deg y_i}$ in place of y_i , and write $R_{\deg h_i}$, $R_{\deg f_j}$, $R_{\deg g_j}$ instead of h_i, f_j, g_j . In view of the map $\psi : G/T \rightarrow BT$ in (3.6) we shall also reserve the nation $c_r(G)$ for the class $\psi^*(c_r(G)) \in H^*(G/T)$.

Theorem 5.1. *The ring $H^*(G/T)$ has the following presentations*

- i) $H^*(SU(n)/T) = \mathbb{Z}[\omega_1, \dots, \omega_{n-1}] / \langle c_2, \dots, c_n \rangle$, $c_r = c_r(SU(n))$,
- ii) $H^*(Sp(n)/T) = \mathbb{Z}[\omega_1, \dots, \omega_n] / \langle c_2, c_4, \dots, c_{2n} \rangle$, $c_{2r} = c_{2r}(Sp(n))$.
- iii) $H^*(E_6/T) = \mathbb{Z}[\omega_1, \dots, \omega_6, x_3, x_4] / \langle R_2, R_3, R_4, R_5, R_6, R_8, R_9, R_{12} \rangle$, where
 - $R_2 = 4\omega_2^2 - c_2$;
 - $R_3 = 2x_3 + 2\omega_2^3 - c_3$;
 - $R_4 = 3x_4 + \omega_2^4 - c_4$;
 - $R_5 = 2\omega_2^2 x_3 - \omega_2 c_4 + c_5$;
 - $R_6 = x_3^2 - \omega_2 c_5 + 2c_6$;
 - $R_8 = x_4(c_4 - \omega_2^4) - 2c_5 x_3 - \omega_2^2 c_6 + \omega_2^3 c_5$;
 - $R_9 = 2x_3 c_6 - \omega_2^3 c_6$;
 - $R_{12} = x_4^3 - c_6^2$.
- iv) $H^*(E_7/T) = \mathbb{Z}[\omega_1, \dots, \omega_7, x_3, x_4, x_5, x_9] / \langle R_t \rangle$, where $t \in \{2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 18\}$, and where
 - $R_2 = 4\omega_2^2 - c_2$;
 - $R_3 = 2x_3 + 2\omega_2^3 - c_3$;
 - $R_4 = 3x_4 + \omega_2^4 - c_4$;
 - $R_5 = 2x_5 - 2\omega_2^2 x_3 + \omega_2 c_4 - c_5$;
 - $R_6 = x_3^2 - \omega_2 c_5 + 2c_6$;
 - $R_8 = 3x_4^2 - x_5(2\omega_2^3 - c_3) - 2x_3 c_5 + 2\omega_2 c_7 - \omega_2^2 c_6 + \omega_2^3 c_5$;
 - $R_9 = 2x_9 + x_4(2\omega_2^2 x_3 - \omega_2 c_4 + c_5) - 2x_3 c_6 - \omega_2^2 c_7 + \omega_2^3 c_6$;
 - $R_{10} = x_5^2 - 2x_3 c_7 + \omega_2^3 c_7$;
 - $R_{12} = x_4^3 - 4x_5 c_7 - c_6^2 + (2\omega_2^3 - c_3)(x_9 + x_4 x_5) + 2\omega_2 x_5 c_6 + 3\omega_2 x_4 c_7 + c_5 c_7$;
 - $R_{14} = c_7^2 - (2\omega_2^2 x_3 - \omega_2 c_4 + c_5)x_9 + 2x_3 x_4 c_7 - \omega_2^3 x_4 c_7$;
 - $R_{18} = x_9^2 + 2x_5 c_6 c_7 - x_4 c_7^2 - (2\omega_2^2 x_3 - \omega_2 c_4 + c_5)x_4 x_9 - (2\omega_2^3 - c_3)x_5^3 - 5\omega_2 x_5^2 c_7$.

As in Section 4.1 we take a set $\Omega = \{\phi_1, \dots, \phi_m\}$ of fundamental dominant weights as a basis for the unit lattice Λ_e of the adjoint Lie group PG , and let $\{t_1, \dots, t_m\}$ be the corresponding basis on the group $H^1(T)$. Granted with the Cartan matrices of simple Lie groups given in [10, p.59] the formula (2.9) for the transgression τ in the fibration $\pi : PG \rightarrow G/T$ yields that

Lemma 5.2. *In the order of $G = SU(n), Sp(n), E_6, E_7$ the following relations hold in the quotient group $H^2(G/T)/\text{Im } \tau$*

- i) $\omega_k = k\omega_1, 1 \leq k \leq n-1, n\omega_1 = 0;$
- ii) $\omega_k = k\omega_1, 1 \leq k \leq n, 2\omega_1 = 0;$
- iii) $\omega_2 = \omega_4 = 0, \omega_1 = \omega_5 = 2\omega_3 = 2\omega_6, 3\omega_1 = 0;$
- iv) $\omega_1 = \omega_3 = \omega_4 = \omega_6 = 0, \omega_5 = \omega_7 = \omega_2, 2\omega_2 = 0.$

Consequently

- a) $c_r(SU(n))|_{\tau(t_1)=\dots=\tau(t_{n-1})=0} = \binom{n}{r}\omega_1^r, 2 \leq r \leq n;$
- b) $c_{2r}(Sp(n))|_{\tau(t_1)=\dots=\tau(t_n)=0} = \binom{n}{r}\omega_1^{2r}, 1 \leq r \leq n;$
- c) $c_r(E_6)|_{\tau(t_1)=\dots=\tau(t_6)=0} = (-1)^r \binom{6}{r}\omega_1^r, 1 \leq r \leq 6;$
- d) $c_r(E_7)|_{\tau(t_1)=\dots=\tau(t_7)=0} = \binom{7}{r}\omega_2^r, 1 \leq r \leq 7. \square$

5.2 Computing with the \mathbb{F}_p -characteristic polynomials

Let $(G, p) = (SU(n), p), (Sp(n), 2), (E_6, 3)$ and $(E_7, 2)$. From Theorem 5.1 one deduces a set $S_p(G) := \{\delta_1, \dots, \delta_m\}$ of primary characteristic polynomials for G over \mathbb{F}_p (see Example 3.7) as that presented in the following table, where $(H)_p \in \mathbb{F}_p[\omega_1, \dots, \omega_m]$ denotes $H \bmod p$, $H \in \mathbb{Z}[\omega_1, \dots, \omega_m]$.

Table 5.1. A set $S_p(G)$ of primary characteristic polynomials for G over \mathbb{F}_p

(G, p)	$S_p(G) = \{\delta_1, \dots, \delta_m\}$
$(SU(n), p)$	$(c_k)_p, 2 \leq k \leq n$
$(Sp(n), 2)$	$(c_{2k})_2, 1 \leq k \leq n$
$(E_6, 3)$	$(\omega_2^2 - c_2)_3, (c_2^2 - c_4)_3, (c_5 + c_2c_3)_3, (c_6 - c_2c_4 - c_3^2)_3$ $(-c_3c_5 - c_2c_6)_3, (c_6c_3)_3$
$(E_7, 2)$	$(c_2)_2, (c_3)_2, (c_5 + \omega_2c_4)_2, (c_4^2 + \omega_2^2c_6 + \omega_2^3c_5 + \omega_2^8)_2,$ $(\omega_2^2c_7 + \omega_2^3c_6)_2, (c_6^2 + c_4^3)_2, (c_7^2 + c_4^2c_6 + \omega_2^2c_6^2)_2$

Let τ' be the transgression in the fibration π' contained in the diagram (3.8). Granted with the class $\varpi = \tau'(t_0)$ determined in Lemma 4.2, as well as the results of Lemma 5.2, formula (3.17) is applicable to evaluate the derivative $\partial P / \partial \varpi$ for $P \in S_p(G)$, as that presented in the following table:

Table 5.2. The derivative $\partial P / \partial \varpi$ for $P \in S_p(G)$

(G, p)	$\{\partial P / \partial \varpi \mid P \in S_p(G)\}$
$(SU(n), p)$	$(\binom{n}{k}\omega_1^{k-1})_p, 2 \leq k \leq n$
$(Sp(n), 2)$	$(\binom{n}{k}\omega_1^{2k-1})_2, 1 \leq k \leq n$
$(E_6, 3)$	$0, 0, 0, 0, 0, (\omega_1^8)_3$
$(E_7, 2)$	$(\omega_2)_2, (\omega_2^2)_2, 0, 0, 0, 0, (\omega_2^{13})_2$

Consequently, one obtains $\theta_1(\varphi_p(P)) \in E_3^{*,0}(PG; \mathbb{F}_p)$, $P \in S_p(G)$, as that stated in the formula (4.5).

By the algorithm given in the proof of Lemma 3.11, for each $P \in S_p(G)$ with $\theta_1(\varphi_p(P)) = 0$ one can construct a polynomial $P' \in \langle \text{Im } \tilde{\tau}_p \rangle \cap \ker f_p$ satisfying the relation $C^*(\varphi_p(P')) = \varphi_p(P)$. Explicitly, a set $S_p(PG)$ of the polynomials P' so obtained is given in the following table:

Table 5.3. A set $S_p(PG)$ of primary characteristic polynomials over \mathbb{F}_p

(G, p)	$S_p(PG) \subset \langle \text{Im } \tilde{\tau}_p \rangle \cap \ker f_p$
$(SU(n), p)$	$(c_k)_p$ for $2 \leq k < p^r$, $(c_k - t_{n,k} c_{p^r} \omega_1^{k-p^r})_p$ for $k \geq p^r$ where $n = p^r n'$ with $(n', p) = 1$;
$(Sp(n), 2)$	$(c_{2k})_2$ for $2 \leq k < 2^r$, $(c_{2k} - t_{n,2} c_{2^{r+1}} \omega_1^{2(k-2^r)})_2$ for $k \geq 2^r$ where $n = 2^r(2b+1)$;
$(E_6, 3)$	$(\omega_2^2 - c_2)_3, (c_2^2 - c_4)_3, (c_5 + c_2 c_3)_3,$ $(c_6 - c_2 c_4 - c_3^2)_3, (-c_3 c_5 - c_2 c_6)_3;$
$(E_7, 2)$	$(c_3 - c_2 \omega_2)_2, (c_5 + \omega_2 c_4)_2, (c_4^2 + \omega_2^2 c_6 + \omega_2^3 c_5 + \omega_2^8)_2,$ $(\omega_2^2 c_7 + \omega_2^3 c_6)_2, (c_6^2 + c_4^3)_2, (c_7^2 + c_4^2 c_6 + \omega_2^2 c_6^2 - c_2 \omega_2^{12})_2$

where $t_{n,k} > 0$ is an integer with $t_{n,k} \binom{n}{p^r} \equiv \binom{n}{k} \pmod{p}$.

By formula (4.7) the cohomology $H^*(PG; \mathbb{F}_p)$ has the presentation

$$(5.2) \quad H^*(PG; \mathbb{F}_p) = \pi^* E_3^{*,0}(PG; \mathbb{F}_p) \otimes \Delta(\iota, \zeta_{2 \deg P-1})_{P \in S_p(PG)},$$

where $\zeta_{2 \deg P-1} = \kappa \circ \varphi_p(P)$, $P \in S_p(PG)$. In the statement and the proof of the following result, we note by the proof of Lemma 3.7 that, with $\zeta_{2 \deg P-1} \in \text{Im } \kappa$,

$$(5.3) \quad \beta_p(\zeta_{2 \deg P-1}) \in \pi^*(E_3^{*,0}(PG)) \subset H^*(PG),$$

where the ring $E_3^{*,0}(PG)$ has been determined by (4.2).

Lemma 5.3. *With respect to (5.2) the Bockstein $\beta_p : H^*(PG; \mathbb{F}_p) \rightarrow H^*(PG)$ satisfies that*

- i) for $(G, p) = (SU(n), p)$ with $n = p^r n'$, $(n', p) = 1$
 $\beta_p(\zeta_{2s-1}) = -p^{r-t-1} \omega^{p^t}$ if $s = p^t$ with $t < r$, 0 otherwise;
- ii) for $(G, p) = (Sp(n), 2)$ with $n = 2^r(2b+1)$:
 $\beta_2(\zeta_{4s-1}) = \omega^{2^r}$ if $s = 2^{r-1}$, 0 if $s \neq 2^{r-1}$;
- iii) for $(G, p) = (E_6, 3)$ and in the order of $s = 2, 4, 5, 6, 8$
 $\beta_3(\zeta_{2s-1}) = 0, -x_4, 0, 0, -x_4^2$;
- iv) for $(G, p) = (E_7, 2)$ and in the order of $s = 3, 5, 8, 9, 12, 14$
 $\beta_2(\zeta_{2s-1}) = x_3, x_5, x_3 x_5, x_9, x_3 x_9, x_5 x_9$.

Proof. For the case $(SU(n), p)$ with $n = p^r n'$ and $(n', p) = 1$ (resp. $(Sp(n), 2)$ with $n = 2^r(2b+1)$), one has by (5.3) that

$$\begin{aligned} \beta_p(\zeta_{2s-1}) &\in \mathbb{Z}[\omega]^+ / \langle p^r \omega, p^{r-1} \omega^p, \dots, \omega^{p^r} \rangle \\ (\text{resp. } \beta_2(\zeta_{4s-1}) &\in \mathbb{Z}[\omega] / \langle 2\omega, \omega^{2^{r+1}} \rangle), \end{aligned}$$

where $\omega = \pi^*(\omega_1)$ (see iii) of Theorem 3.9). By the degree reason one gets

$$\beta_p(\zeta_{2s-1}) = 0 \text{ if } s \geq p^r \text{ (resp. } \beta_2(\zeta_{4s-1}) = 0 \text{ if } s \geq 2^r\text{).}$$

In the remaining cases $2 \leq s < p^r$ (resp. $1 \leq s < 2^r$) an integral lift of the characteristic polynomial $(c_s)_p$ (resp. $(c_{2s})_2$) of the class ζ_{2s-1} (resp. ζ_{4s-1}) is easily seen to be

$$c_s - \binom{n}{s} \omega_1^s \in \langle \text{Im } \tilde{\tau} \rangle \text{ (resp. } c_{2s} - \binom{n}{k} \omega_1^{2s} \in \langle \text{Im } \tilde{\tau} \rangle\text{).}$$

The formula in Lemma 3.7 then yields that

$$\begin{aligned} \beta_p(\zeta_{2s-1}) &= \pi^* \frac{1}{p} f(c_s - \binom{n}{s} \omega_1^s) = \pi^* \left(-\frac{1}{p} \binom{n}{s} \omega_1^s \right) = -\frac{1}{p} \binom{n}{s} \omega^s \\ (\text{resp. } \beta_2(\zeta_{4s-1}) &= \pi^* \left(\frac{1}{2} f(c_{2s} - \binom{n}{s} \omega_1^{2s}) \right) = \pi^* \left(\frac{1}{2} \binom{n}{s} \omega_1^{2s} \right) = \frac{1}{2} \binom{n}{s} \omega^{2s}), \end{aligned}$$

where the second equality follows from $f(c_s) = 0$ (resp. $f(c_{2s}) = 0$) by i) (resp. ii) of Theorem 5.1. The relation i) (resp. ii) of the present lemma comes now from $p^{r-t} \omega^{p^t} = 0$, $0 \leq t \leq r$ (resp. $2\omega, \omega^{2^{r+1}} = 0$), and the property (5.6) of the binomial coefficients $\binom{n}{s}$.

Turning to the cases $(G, p) = (E_6, 3)$ or $(E_7, 2)$ for each $P = (H)_p \in S_p(PG)$ given by Table 5.3 the enclosed polynomial H satisfies $H \in \langle \text{Im } \tilde{\tau} \rangle$ by the relation c) or d) of Lemma 5.2. Therefore, by Lemma 3.7

$$\beta_p(\zeta_{2 \deg P - 1}) = \pi^* \left(\frac{1}{p} f(H) \mid_{\tau(t_1) = \dots = \tau(t_n) = 0} \right).$$

This formula, together with the relations R_i 's in the presentations iii) and iv) of Theorem 5.1, suffices to yield the results in iii) and iv). As examples, when $(G, p) = (E_6, 3)$ it gives rise to

$$\begin{aligned} \beta_3(\zeta_3) &= \pi^* \frac{\omega_2^2 - c_2}{3} = \pi^* \omega_2^2 = 0; \\ \beta_3(\zeta_7) &= \pi^* \frac{c_2^2 - c_4}{3} = \pi^* (5\omega_2^2 - x_4) = -x_4; \\ \beta_3(\zeta_9) &= \pi^* \frac{c_5 + c_2 c_3}{3} = \pi^* (\omega_2^5 + \omega_2 x_4 + \omega_2^2 c_3) = 0; \\ \beta_3(\zeta_{11}) &= \pi^* \frac{c_2 c_4 + c_3^2 - c_6}{3} = \pi^* (5\omega_2^2 x_4 + \omega_2^6 + \omega_2 c_5 + \omega_2^3 c_3 - 3c_6) = 0; \\ \beta_3(\zeta_{15}) &= \pi^* \left(-\frac{c_3 c_5 + c_2 c_6}{3} \right) = \pi^* (-4x_4^2 + c_3 c_5 + 4\omega_2^3 c_5) = -x_4^2, \end{aligned}$$

where, in each of the above equations, the second equality is deduced from the relations R_i 's in iii) of Theorem 5.1, where the last one is obtained by the formula c) of Lemma 5.2. \square

For $G = SU(n), Sp(n), E_6$ or E_7 consider the fibration $G/T \xrightarrow{\psi} BT \xrightarrow{B_i} BG$ induced by the inclusion $i : T \rightarrow G$ of a maximal torus T , see (3.6). It was shown in [8, Lemma 5.4] that there exists a complex bundle ξ on BT so that

$$\begin{aligned} C(\xi) &= 1 + c_1 + c_2 + \dots \text{ for } G = SU(n), E_6 \text{ or } E_7; \\ P(\xi) &= 1 + c_2 + c_4 + \dots \text{ for } G = Sp(n), \end{aligned}$$

where $C(\xi)$ and $P(\xi)$ are the total Chern class and the total Pontrjagin class of ξ , respectively. It follows from the Wu-formula [17, p.94] that the following relation holds on the ring $H^*(BT; \mathbb{F}_2)$

(5.4) $Sq^{2s-2}c_s = c_{s-1}c_s + c_{s-2}c_{s+1} + \cdots + c_{2s-1}$ for $G = SU(n), E_6$ or E_7 ;
 $Sq^{4s-2}c_{2s} = c_{2s-2}c_{2s} + c_{2s-4}c_{2s+2} + \cdots + c_{4s-2}$ for $G = Sp(n)$.

Lemma 5.4. *With respect to the presentation (5.2) the Steenrod operators δ_2 and Sq^{2r} on $H^*(PG; \mathbb{F}_p)$ satisfy the following relations*

- i) $G = SU(n)$ with $n = 2^r(2b + 1)$:
 - $\delta_2\zeta_{2s-1} = \omega_1^{2^{r-1}}$ if $s = 2^{r-1}$, 0 if $s \neq 2^{r-1}$;
 - $Sq^{2s-2}\zeta_{2s-1} = \zeta_{4s-3}$ for $2s - 1 \leq 2^{r-1}$;
- ii) $G = Sp(n)$ with $n = 2^r(2b + 1)$:
 - $\delta_2\zeta_{4s-1} = \omega_1^{2^r}$ if $s = 2^{r-1}$, 0 if $s \neq 2^{r-1}$;
 - $Sq^{4s-2}\zeta_{4s-1} = \zeta_{8s-3}$ for $4s - 1 \leq 2^r$;
- iii) $G = E_7$ and in accordance to $s = 3, 5, 8, 9, 12, 14$:
 - $\delta_2\zeta_{2s-1} = y_3, y_5, y_3y_5, y_9, y_3y_9, y_5y_9$;
 - $Sq^{2s-2}\zeta_{2s-1} = \zeta_9, \zeta_{17}, 0, 0, 0, 0$.

Proof. The results on $\delta_2\zeta_{2s-1}$ comes directly from the relation $\delta_2 = r_2 \circ \beta_2$, in which the action of β_2 on ζ_{2s-1} has been decided in Lemma 5.3, while the reduction r_2 is transparent in view of the presentations of the rings $E_3^{*,0}(PG)$ and $E_3^{*,0}(PG; \mathbb{F}_2)$ in (4.2) and Lemma 4.3.

For $G = SU(n)$ (resp. $Sp(n)$) with $n = 2^r(2b + 1)$ we get from Lemma 3.8 and the formula (5.4) that

$$Sq^{2s-2}\zeta_{2s-1} = \kappa\varphi_2(Sq^{2s-2}c_s) = \kappa\varphi_2(c_{2s-1}) = \zeta_{4s-3}$$

(resp. $Sq^{4s-2}\zeta_{4s-1} = \varphi_2(Sq^{4s-2}c_{2s}) = \varphi_2(c_{4s-2}) = \zeta_{4s-3}$),

where $2s - 1 \leq 2^{r-1}$ (resp. $4s - 1 \leq 2^r$), the second equality comes from $\varphi_2(c_{s-1-t}c_{s+t}) = 0$ for all $0 \leq t \leq s - 2$ by Lemma 3.4, and where the last equality follows from the fact that, with $2s - 1 < 2^{r-1}$, c_{2s-1} is a characteristic polynomial of the class $\zeta_{4s-3} \in H^*(PSU(n); \mathbb{F}_2)$, see Table (5.3). This completes the proof of i) (resp. ii)).

Finally, for the case $G = E_7$, granted with the set $S_2(PE_7)$ of explicitly polynomials given by Table 5.3 and the Wu-formula (5.4), it is straightforward to show the formulae in iii) by the method entailed above. Alternatively, see [8, Section 5.2] for the relevant computation. \square

5.3 The integral characteristic polynomials

In view of the presentation of the rings $H^*(G/T)$ in Theorem 5.1, one formulates the set $S(G)$ of characteristic polynomials for the 1-connected groups $G = SU(n), Sp(n), E_6, E_7$ (see Example 3.6) as that tabulated below

Table 5.4. The set $S(G)$ of primary characteristic polynomials over \mathbb{Z}

G	$S(G)$
$SU(n)$	$c_k, 2 \leq k \leq n$
$Sp(n)$	$c_{2k}, 1 \leq k \leq n$
E_6	$R_2, R_5, 2R_6 - x_3R_3, R_8, R_9, 3R_{12} - x_4^2R_4$
E_7	$R_2, 2R_6 - x_3R_3, R_8, 2R_{10} - x_5R_5, 3R_{12} - x_4^2R_4, R_{14}, 2R_{18} - x_9R_9$

In (4.9), (4.17a) and (4.17b), the rings $H^*(G)$ are presented by the primary 1-forms $\gamma_{2 \deg P-1} := \kappa\varphi(P) \in E_3^{*,1}(G)$ with $P \in S(G)$.

Let τ' be the transgression in the fibration π' in the diagram (3.8). Granted with the class $\varpi = \tau'(t_0)$ determined by Lemma 4.2, as well as the results of Lemma 5.2, formula (3.17) is ready to apply to evaluate the derivation $\partial P / \partial \varpi$ for $P \in S(G)$. The results are so obtained presented in the following table.

Table 5.5. The derivation $\partial P / \partial \varpi$ for $P \in S(G)$

G	$\{\partial P / \partial \varpi \mid P \in S(G)\}$
$SU(n)$	$\binom{n}{s} \omega_1^{s-1}, 2 \leq s \leq n$
$Sp(n)$	$\binom{n}{s} \omega_1^{2s-1}, 1 \leq s \leq n$
E_6	$0, 0, 0, 0, \omega_1^8, 0$
E_7	$\omega_2, \omega_2^2 y_3, \omega_2^2 y_5, 0, \omega_2^2 (y_9 + y_4 y_5 + y_4 \omega_2^5), \omega_2^9 (y_4 + \omega_2^4), 0$

5.4 Arithmetic properties of binomial coefficients

We begin with a brief account for three arithmetic properties of the binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ which are required by computing with cohomology of the adjoint Lie group $PSU(n)$, where $1 \leq k \leq n$.

Consider $b_{n,k} = g.c.d.\{\binom{n}{1}, \dots, \binom{n}{k}\}$, $1 \leq k \leq n$. Since $b_{n,k+1} \mid b_{n,k}$ with $b_{n,1} = n, b_{n,n} = 1$ one defines the integers $a_{n,k} := \frac{b_{n,k-1}}{b_{n,k}}, k \geq 2$. The following result has been shown in [4], and is required by proving i) of Lemma 4.6.

(5.5) If $n > 2$ is an integer with the prime factorization $n = p_1^{r_1} \cdots p_t^{r_t}$, then $a_{n,k} = p_i$ or 1 in accordance to $k \in Q_{p_i}(n)$ or $k \in Q_0(n)$, where $1 \leq i \leq t$.

For an integer m denote by $ord_p m$ the biggest integer a so that m is divisible by the power p^a . The following result is needed by proving i) of Lemma 5.3. Assume that $n = p^r n'$ with $(n', p) = 1$.

(5.6) For any $p^t \leq s < p^{t+1}$ with $t+1 < r$ one has $ord_p \binom{n}{s} \geq r-t$, while the equality holds iff $s = p^t$.

The remaining part of this section is devoted to a proof of Proposition 4.8. The calculation will be based on the following result

(5.7) For any $1 \leq s \leq r$ there is a sequence $\{h_1, \dots, h_s\}$ of integers such that

$$\binom{p^r}{p^s} - p^{r-s} = h_1 \binom{p^r}{p^{s-1}} + h_2 \binom{p^r}{p^{s-2}} + \cdots + h_s \binom{p^r}{1}.$$

The center of the special unitary group $SU(n)$ is the cyclic group \mathbb{Z}_n generated by the diagonal matrix $diag\{e^{\frac{2\pi i}{n}}, \dots, e^{\frac{2\pi i}{n}}\} \in SU(n)$. The total space of the circle bundle C on $PSU(n)$ (see (1.4)) is the unitary group $U(n)$, whose maximal torus T is $\{diag\{e^{i\theta_1}, \dots, e^{i\theta_n}\} \mid \theta_i \in [0, 2\pi]\}$. In the terminologies of Section 3.4 we can furnish the group $H^1(T)$ with the basis $\{t_1, \dots, t_{n-1}, t_0\}$ in which the subset $\{t_1, \dots, t_{n-1}\}$ corresponds to a set of fundamental dominant weights of the group $PSU(n)$. Consequently, in the short exact sequence (3.9) associated to the bundle C

$$0 \rightarrow E_2^{*,k}(PSU(n)) \xrightarrow{C_2^*} E_2^{*,k}(U(n)) \xrightarrow{\overline{\theta}} E_2^{*,k-1}(PSU(n)) \rightarrow 0$$

one has

$$E_2^{*,k}(PSU(n)) = H^*(\frac{U(n)}{T}) \otimes \Lambda(t_1, \dots, t_{n-1});$$

$$(\text{resp. } E_2^{*,k}(U(n)) = H^*(\frac{U(n)}{T}) \otimes \Lambda(t_1, \dots, t_{n-1}, t_0)),$$

while by ii) of Remark 2.5 the d_2 actions on $E_2^{*,k}$ are determined by

$$\begin{aligned} \tau(t_1) &= 2\omega_1 - \omega_2; \tau(t_2) = -\omega_1 + 2\omega_2 - \omega_3; \dots; \\ \tau(t_{n-1}) &= -\omega_{n-2} + 2\omega_{n-1} \text{ and } \tau'(t_0) = \omega_1 \end{aligned}$$

where τ and τ' are the transgression in π and π' (see in the diagram (3.8)), respectively. It follows that if $\{c_2, \dots, c_n\}$ is the set of primary characteristic polynomials for the group $SU(n)$ over \mathbb{Z} given by Table 5.4, then

$$H^*(U(n)) = E_3^{*,*}(U(n)) = \Lambda(\gamma_1, \gamma_3, \dots, \gamma_{2n-1}),$$

where, in terms of Lemmas 3.2 and 3.3,

$$\begin{aligned} \gamma_1 &= -(n-1) \otimes t_1 - (n-2) \otimes t_2 - \dots - 1 \otimes t_{n-1} + n \otimes t_0, \\ \gamma_{2r-1} &= [\tilde{c}_r]. \end{aligned}$$

In the following result we assume that $n = p^r$ with p a prime.

Lemma 5.5. *For each $p^s \in Q_p(n)$ the class γ_{2p^s-1} has a characteristic polynomial of the form $\alpha_{p^s} + p^{r-s}\omega_1^{p^s}$ with $\alpha_{p^s} \in \langle \text{Im } \tilde{\tau} \rangle$. As results*

- i) $\gamma_{2p^s-1} = [\tilde{\alpha}_{p^s} + p^{r-s}\omega_1^{p^s-1} \otimes t_0]$ with $\tilde{\alpha}_{p^s} \in E_2^{*,1}(PSU(n))$;
- ii) $d_2(\tilde{\alpha}_{p^s}) = -p^{r-s}\omega_1^{p^s}$.

Proof. For a given $p^s \in Q_p(n)$ let $\{h_1, \dots, h_s\}$ be the sequence of integers satisfying (5.7). Consider the polynomial

$$\alpha_{p^s} = c_{p^s} - h_1\omega^{p^s-p^{s-1}}c_{p^{s-1}} - \dots - h_s\omega^{p^s-1}c_1 - p^{r-s}\omega_1^{p^s} \in \mathbb{Z}[\omega_1, \dots, \omega_{n-1}]$$

From $\alpha_{p^s} |_{\tau(t_1) = \dots = \tau(t_{n-1}) = 0} = 0$ by (5.7) and a) of Lemma 5.2 one finds that $\alpha_{p^s} \in \langle \text{Im } \tilde{\tau} \rangle$. Moreover, with respect to the surjection

$$f : \mathbb{Z}[\omega_1, \dots, \omega_{n-1}] \rightarrow H^*(U(n)/T) \text{ (see Section 3.2)}$$

the polynomial $\alpha_{p^s} + p^{r-s}\omega_1^{p^s} \in \mathbb{Z}[\omega_1, \dots, \omega_{n-1}]$ has a lift to $E_2^{*,1}(U(n))$ (see (3.1)) with the form

$$\begin{aligned} &\tilde{\alpha}_{p^s} - h_1\omega^{p^s-p^{s-1}}\tilde{c}_{p^{s-1}} - h_2\omega^{p^s-p^{s-2}}\tilde{c}_{p^{s-2}} - \dots - h_s\omega^{p^s-1}\tilde{c}_1 \\ &= \tilde{\alpha}_{p^s} - d_2(h_1\omega^{p^s-p^{s-1}-1}\tilde{c}_{p^{s-1}} + h_2\omega^{p^s-p^{s-2}-1}\tilde{c}_{p^{s-2}} + \dots + h_s\omega^{p^s-2}\tilde{c}_1) \otimes t_0. \end{aligned}$$

These shows that $\gamma_{2p^s-1} = [\tilde{\alpha}_{p^s} + p^{r-s}\omega_1^{p^s-1} \otimes t_0]$ with $\tilde{\alpha}_{p^s} \in E_2^{*,1}(PSU(n))$. \square

By ii) of Lemma 5.5 the 1-forms $p \cdot \tilde{\alpha}_{p^s} - \omega^{p^s-p^{s-1}}\tilde{\alpha}_{p^s-1} \in E_2^{*,1}(PSU(n))$ with $1 \leq s \leq r$ are d_2 closed and therefore, define the cohomology classes

$$(5.8) \quad \rho_{2p^s-1} := \left[p\tilde{\alpha}_{p^s} - \omega^{p^s-p^{s-1}}\tilde{\alpha}_{p^{s-1}} \right] \in E_3^{*,1}(PSU(n)), \quad 1 \leq s \leq r.$$

Moreover, the computation

$$\begin{aligned} & C^*(\rho_{2p^s-1}) - p \cdot \gamma_{2p^s-1} \\ &= \left[p\tilde{\alpha}_{p^s} - \omega^{p^s-p^{s-1}}\tilde{\alpha}_{p^{s-1}} \right] - \left[p\tilde{\alpha}_{p^s} + p^{r-s+1}\omega_1^{p^s-1} \otimes t_0 \right] \\ &= \left[-\omega^{p^s-p^{s-1}}\tilde{\alpha}_{p^{s-1}} - p^{r-s+1}\omega_1^{p^s-1} \otimes t_0 \right] \\ &= \left[d_2(-\omega^{p^s-p^{s-1}-1}\tilde{\alpha}_{p^{s-1}} \otimes t_0) \right] = 0 \end{aligned}$$

indicates that $C^*(\rho_{2p^s-1}) = p \cdot \gamma_{2p^s-1}$, where the second equality comes from i) of Lemma 5.5, while the fourth one follows from ii) of Lemma 5.5 and $d_2(t_0) = \omega_1$. Granted with the classes ρ_{2p^s-1} ($1 \leq s \leq r$) defined by (5.8) we are ready to show Proposition 4.8.

Proof of Proposition 4.8. In term of the formulae (4.9) and (4.3) of the rings $H^*(U(n))$ and $H^*(U(n); \mathbb{F}_p)$ the reduction r_p on $H^*(U(n))$ satisfies the relation $r_p(\gamma_I) = \xi_I$. Moreover, if $I = (p^{i_1}, \dots, p^{i_k}) \in Q_p(n)$ with $1 \leq i_1 < \dots < i_k < r$ one has $\theta(\xi_I) = 0$ by (4.5). The commutative diagram induced by r_p

$$\begin{array}{ccc} H^*(U(n)) & \xrightarrow{\theta} & H^*(PSU(n)) \\ r_p \downarrow & & r_p \downarrow \\ H^*(U(n); \mathbb{F}_p) & \xrightarrow{\theta} & H^*(PSU(n); \mathbb{F}_p) \end{array}.$$

now concludes that $r_p(\theta(\gamma_I)) = 0$. This shows assertion i) of Proposition 4.8.

With $\gamma_{2p^s-1} = \left[\tilde{\alpha}_{p^s} + p^{r-s}\omega_1^{p^s-1} \otimes t_0 \right]$ by i) of Lemma 5.5 one gets by (3.10) that

$$\theta(\gamma_{2p^s-1}) = \left[\bar{\theta}(\tilde{\alpha}_{p^s} + p^{r-s}\omega_1^{p^s-1} \otimes t_0) \right] = p^{r-s}\omega^{p^s-1},$$

where we have made use of the relations $\omega_1 = \varpi$ by Lemma 4.2, and $\pi^*(\varpi) = \omega$ by iii) of Theorem 3.9. This shows formula ii) of Proposition 4.8.

Finally, for the formula iii) of Proposition 4.8 one gets from (3.10), the decomposition $\gamma_I = \gamma_{I^e}[\tilde{\alpha}_{p^{i_k}} + p^{r-i_k}\omega_1^{p^{i_k}-1} \otimes t_0]$ by i) of Lemma 5.5, as well as the relation $t_0^2 = 0$ on $E_2^{*,*}$, the following relation in $E_3^{*,*}(PSU(n))$

$$(5.9) \quad \bar{\theta}(\gamma_I) = \bar{\theta}(\gamma_{I^e})\tilde{a}_{p^{i_k}} + p^{r-i_k}\omega^{p^{i_k}-1}\tilde{a}_{I^e},$$

where $\tilde{a}_K = \prod_{s \in K} \tilde{a}_s$. Granted with the fact that the class $\bar{\theta}(\gamma_{I^e})$ is divisible by p the formula iii) is verified by the following calculation

$$\begin{aligned} \bar{\theta}(\gamma_I) &= \left(\frac{1}{p}\theta(\gamma_{I^e}) \right) \cdot p\tilde{a}_{p^{i_k}} + p^{r-i_k}\omega^{p^{i_k}-1}\tilde{a}_{I^e} \\ &= \frac{1}{p}\theta(\gamma_{I^e})(\rho_{2p^{i_k}-1} + \omega^{p^{i_k}-p^{i_k-1}}\tilde{a}_{p^{i_k-1}}) + p^{r-i_k}\omega^{p^{i_k}-1}\tilde{a}_{I^e} \quad (\text{by (5.8)}) \\ &= \frac{1}{p}\theta(\gamma_{I^e})\rho_{2p^{i_k}-1} + \frac{1}{p}\omega^{p^{i_k}-p^{i_k-1}}(\theta(\gamma_{I^e})\tilde{a}_{p^{i_k-1}} + p^{r-i_k+1}\omega^{p^{i_k}-1}\tilde{a}_{I^e}) \\ &= \frac{1}{p}\theta(\gamma_{I^e})\rho_{2p^{i_k}-1} + \frac{1}{p}\omega^{p^{i_k}-p^{i_k-1}}\theta(\gamma_{I^e}) \quad (\text{by (5.9)}). \square \end{aligned}$$

References

- [1] P. F. Baum, W. Browder, The cohomology of quotients of classical groups. *Topology* 3(1965), 305–336.
- [2] C. Chevalley, Sur les Décompositions Cellulaires des Espaces G/B , in Algebraic groups and their generalizations: Classical methods, W. Haboush ed. Proc. Symp. in Pure Math. 56 (part 1) (1994), 1-26.
- [3] J. Dieudonné, A History of Algebraic and Differential Topology 1900–1960, Boston; Basel; Birkhäuser, 1989.
- [4] H. Duan and X. Lin, Topology of unitary groups and the prime orders of binomial coefficients, arXiv:1502.00401.
- [5] H. Duan, SL. Liu, The isomorphism type of the centralizer of an element in a Lie group, *Journal of algebra*, 376(2013), 25-45.
- [6] H. Duan, Xuezhi Zhao, Schubert presentation of the integral cohomology ring of the flag manifolds G/T , *LMS J. Comput. Math.* Vol.18, no.1(2015), 489-506.
- [7] H. Duan, Xuezhi Zhao, Schubert calculus and cohomology of Lie groups. Part I. 1-connected Lie groups, *math.AT* (math.AG). arXiv:0711.2541.
- [8] H. Duan, Xuezhi Zhao, Schubert calculus and the Hopf algebra structures of exceptional Lie groups, *Forum. Math.* Vol.25, no.1(2014), 113–140.
- [9] A. Grothendieck, Torsion homologique et sections rationnelles, Sem. C. Chevalley, ENS 1958, exposé 5, Secreatariat Math. IHP, Paris, 1958.
- [10] J. E. Humphreys, Introduction to Lie algebras and representation theory, Graduated Texts in Math. 9, Springer-Verlag New York, 1972.
- [11] K. Ishitoya, A. Kono, H. Toda, Hopf algebra structure of mod 2 cohomology of simple Lie groups. *Publ. Res. Inst. Math. Sci.* 12 (1976/77), no. 1, 141–167.
- [12] V.G. Kač, Torsion in cohomology of compact Lie groups and Chow rings of reductive algebraic groups, *Invent. Math.* 80(1985), no. 1, 69–79.
- [13] A. Kono, Hopf algebra structure of simple Lie groups, *J. Math. Kyoto Univ.* 17 (1977), no. 2, 259–298.
- [14] W. S. Massey, On the cohomology ring of a sphere bundle, *J. Math. Mech.* 7 1958 265-289.
- [15] J. McCleary, A user’s guide to spectral sequences, Second edition. Cambridge Studies in Advanced Mathematics, 58. Cambridge University Press, Cambridge, 2001.
- [16] R. Marlin, Une conjecture sur les anneaux de Chow $A(G, \mathbb{Z})$ renforcée par un calcul formel, Effective methods in algebraic geometry (Castiglioncello, 1990), 299–311, *Progr. Math.*, 94, Birkhäuser Boston, Boston, MA, 1991.

- [17] J. W. Milnor and J. D. Stasheff, Characteristic classes. Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974.
- [18] M. Mimura and H. Toda, Topology of Lie groups. I, II., Translations of Mathematical Monographs, 91, American Mathematical Society, Providence, RI, 1991.
- [19] M. Reeder, On the cohomology of compact Lie groups. *Enseign. Math.* (2) 41 (1995), no. 3-4, 181–200.
- [20] C. A. Ruiz, The cohomology of the complex projective Stiefel manifold, *Trans. Amer. Math. Soc.* 146(1969), 541–547.
- [21] M. R. Sepanski, Compact Lie groups, Graduate Texts in Mathematics, 235. Springer, New York, 2007.
- [22] N. E. Steenrod and D. B. A. Epstein, Cohomology Operations, Ann. of Math. Stud., Princeton Univ. Press, Princeton, NJ, 1962.
- [23] A. Weil, Foundations of algebraic geometry, American Mathematical Society, Providence, R.I. 1962.